May 29, 2012

海面波によるVortex Force:鉛直幅で重み付け平均された運動方程式

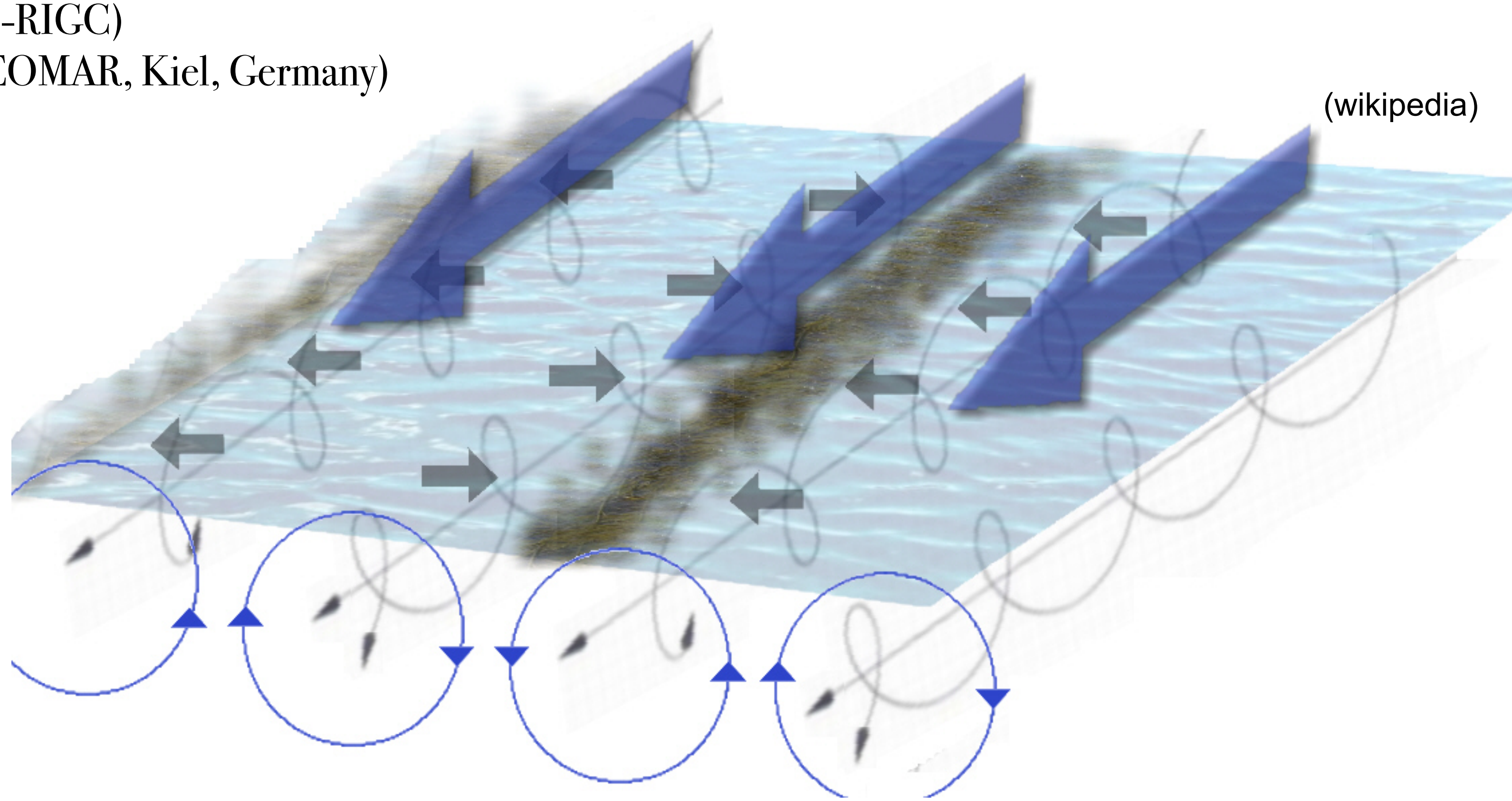
The Vortex Force in the thickness-weighted mean momentum equations for surface gravity waves

Hidenori Aiki (JAMSTEC-RIGC)
Richard J. Greatbatch (GEOMAR, Kiel, Germany)

to be submitted

(wikipedia)

Langmuir
Circulation



Craik and Leibovich momentum equations

Vortex Force

$$(\partial_T + \overline{\mathbf{U}}^E \cdot \nabla)\overline{\mathbf{U}}^E + (\nabla \times \overline{\mathbf{U}}^E) \times \mathbf{U}^S = -\nabla \pi$$

Equations in density-coordinates

Andrews, 1983, JAS
Greatbatch, 1998, JPO

momentum

$$\rho_o(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + f\mathbf{z}\times\mathbf{u}) = -\nabla^c p$$

thickness

$$z_{\rho t} + \nabla\cdot(z_\rho\mathbf{u}) = 0$$

pressure

$$p_z = -g\rho$$

$$z = \bar{z} + z',\ \ p = \bar{p} + p',\ \ \mathbf{u} = \underbrace{\frac{\overline{z_\rho\mathbf{u}}}{\bar{z}_\rho}}_{\hat{\mathbf{u}}} + \mathbf{u}''$$

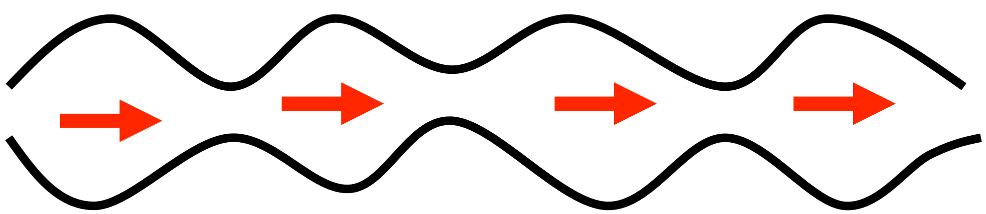
thickness-weighted-mean momentum

$$\rho_o[(\bar{z}_\rho\hat{\mathbf{u}})_t + \nabla\cdot(\bar{z}_\rho\hat{\mathbf{u}}\hat{\mathbf{u}}) + f\mathbf{z}\times\bar{z}_\rho\hat{\mathbf{u}}] =$$
$$-\bar{z}_\rho\nabla^c\bar{p} \underline{- (\overline{z'\nabla^c p'})_\rho + \nabla(g\overline{z'^2}/2)} - \nabla\cdot(\overline{z_\rho\mathbf{u}''\mathbf{u}''})$$

$$\widehat{\mathbf{V}} = \frac{\overline{z_\rho \mathbf{V}}}{\overline{z}_\rho} = \frac{\overline{(\overline{z}_\rho + z'_\rho)(\overline{\mathbf{V}} + \mathbf{V}')}}{\overline{z}_\rho} = \overline{\mathbf{V}} + \frac{\overline{z'_\rho \mathbf{V}'}}{\overline{z}_\rho}$$

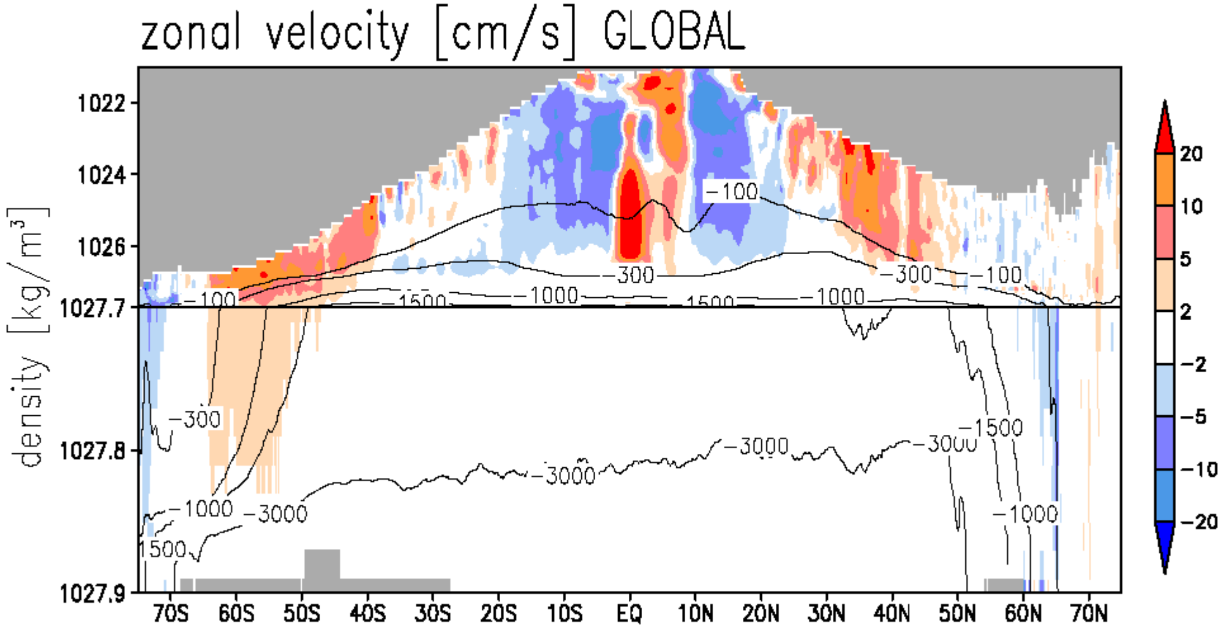
unweighted mean velocity (geostrophic flow, Ekman flow)

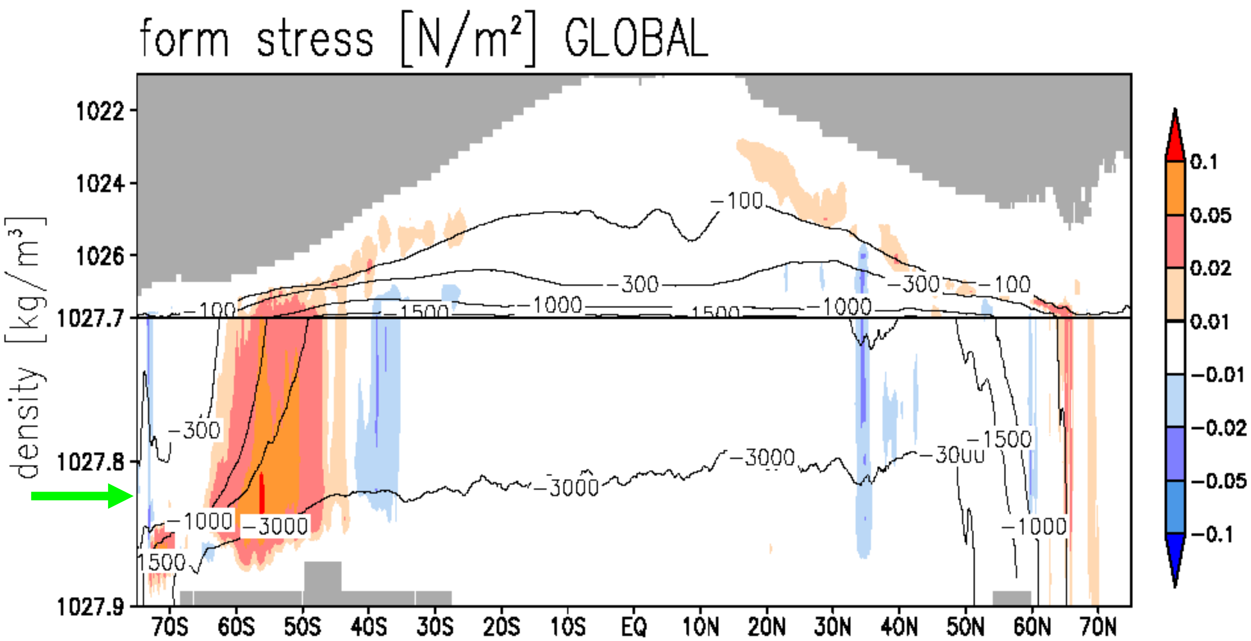
Bolus velocity (eddy-induced overturning circulation)



Rhines, 1982

ZONAL MEAN OF GLOBAL OCEAN

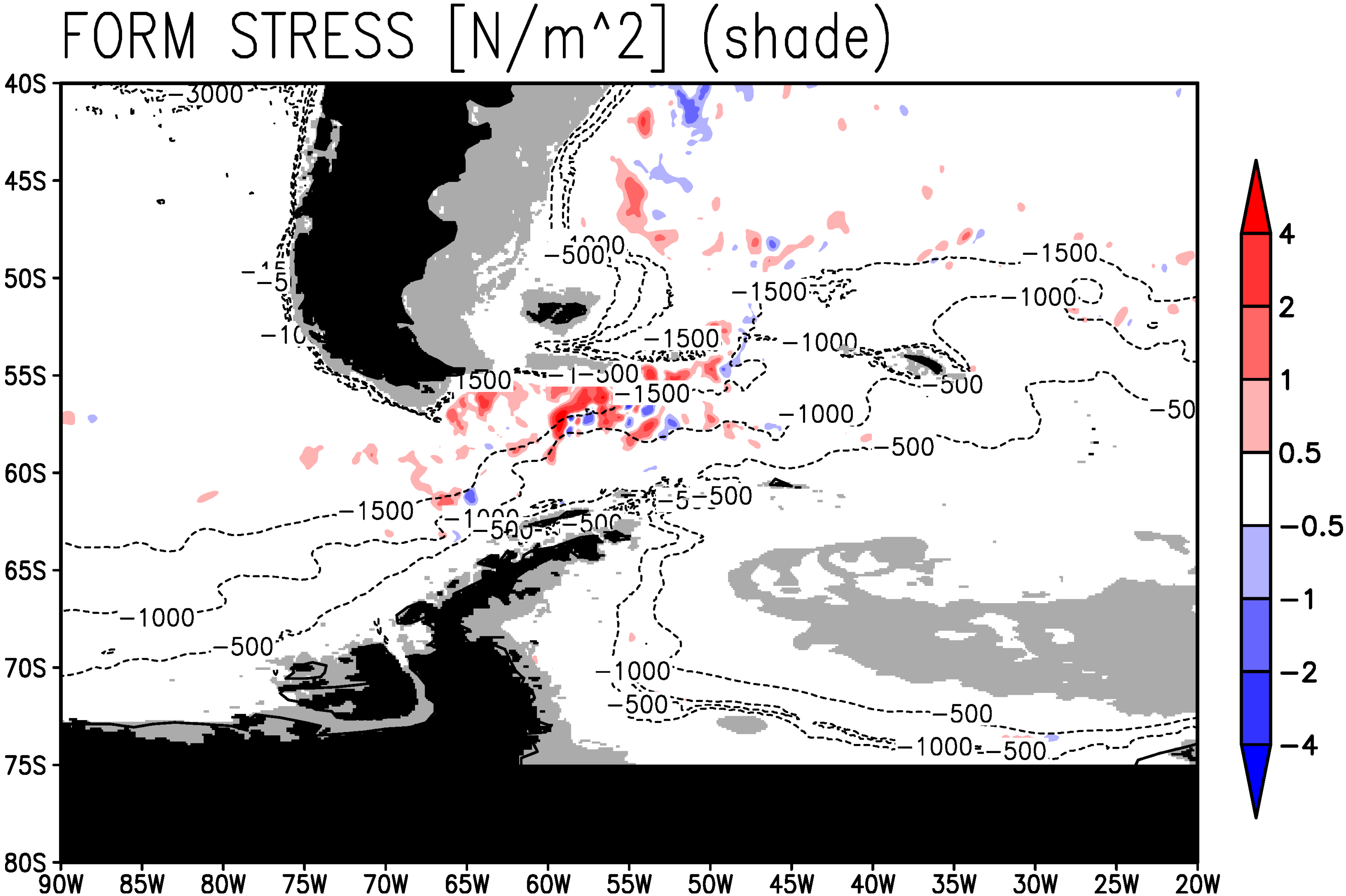




$$-\left(\underline{\overline{z'\nabla^c p'}}\right)_\rho + \nabla\left(g\overline{z'^2}/2\right)$$

Aiki and Richards, 2008, JPO

DRAKE PASSAGE



Aiki and Richards, 2008, JPO

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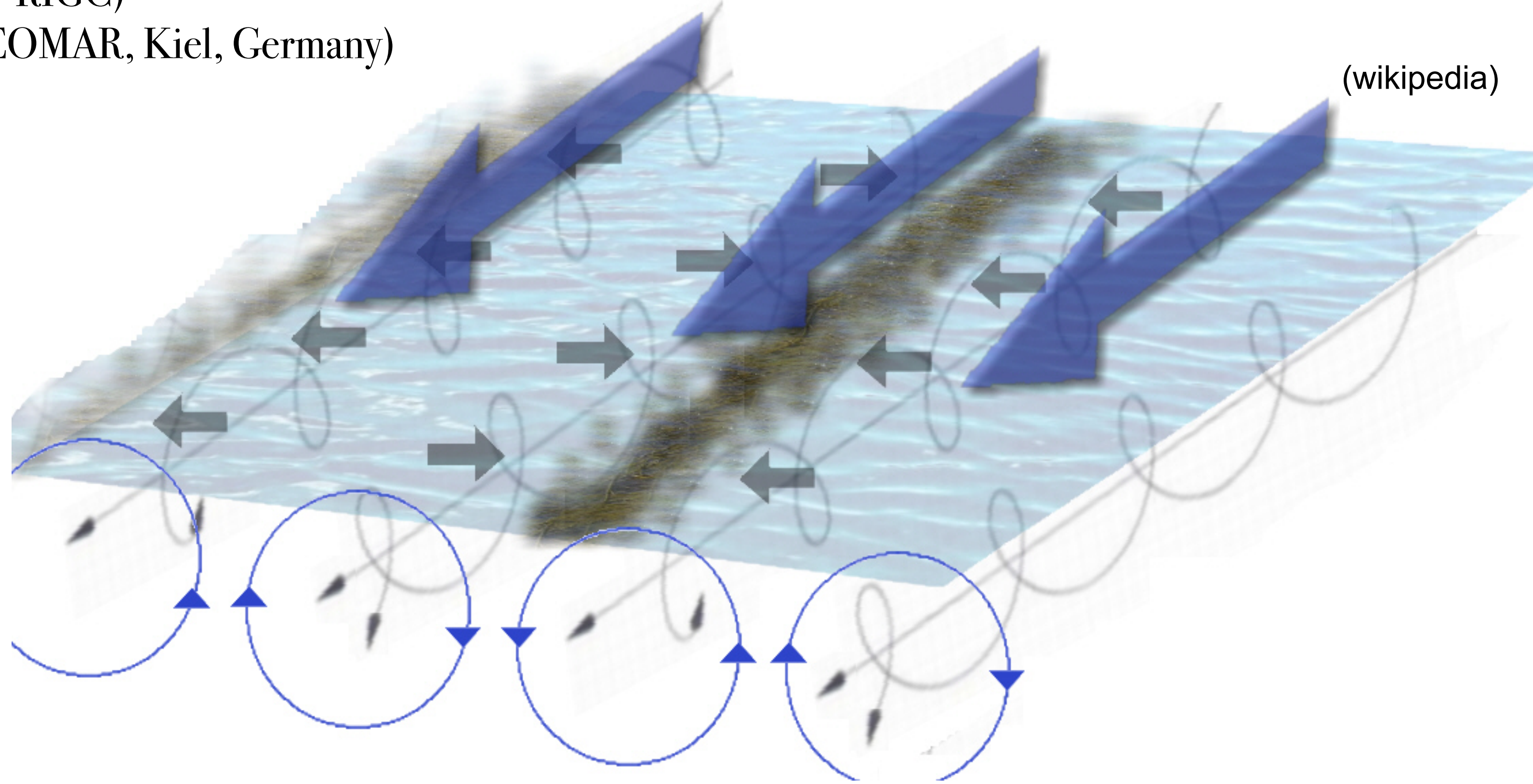

Craik and Leibovich momentum equations

Vortex Force

$$(\partial_T + \overline{\mathbf{U}}^E \cdot \nabla)\overline{\mathbf{U}}^E + (\nabla \times \overline{\mathbf{U}}^E) \times \mathbf{U}^S = -\nabla\pi$$

Studies for the Effect of Surface Waves on Circulation

Vertical	Horizontal	**Radiation Stress** (Reynolds Stress + Form Stress)	**Vortex Force**
Integrated	Eulerian	Longuet-Higgins & Stewart (1964)	Garrett (1976)
Eulerian	Eulerian	-	**Craik & Leibovich (1976)** McWilliams et al. (2004)
Lagrangian	Lagrangian	-	Andrews & McIntyre (1978) **Leibovich (1980)** Ardhuin et al. (2008)
Lagrangian	Eulerian	Mellor (2003), Aiki & Greatbatch (2012)	this study

3D Eulerian framework

(Craik and Leibovich, 1976)

Vertical	Horizontal	Radiation Stress (Reynolds Stress + Form Stress)	Vortex Force
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Lagrangian	Eulerian	Mellor (2003), Aiki & Greatbatch (2012)	this study

3D Eulerian framework (Craik & Leibovich, 1976)

$$\partial_t \mathbf{U}_1' = -\nabla p_1' \qquad\qquad \overline{\mathbf{U}}_1 = 0$$

$$\partial_t \mathbf{U}_2' + \mathbf{U}_1' \bullet \nabla \mathbf{U}_1' = -\nabla p_1'$$

$$\partial_t \mathbf{U}_3' + \mathbf{U}_1' \bullet \nabla \left(\overline{\mathbf{U}}_2 + \mathbf{U}_2' \right) + \left(\overline{\mathbf{U}}_2 + \mathbf{U}_2' \right) \bullet \nabla \mathbf{U}_1' = -\nabla p_3'$$

Time scale of Langmuir Circulations is two orders slower than wave period

$$\partial_T \sim \alpha^2 \partial_t$$

$$\partial_T \overline{\mathbf{U}}_2 + \nabla \bullet \left(\overline{\mathbf{U}}_2 \overline{\mathbf{U}}_2 \right) + \nabla \bullet \left(\overline{\mathbf{U}'\mathbf{U}'} \right)_4 = -\nabla \overline{p}_4$$

$$\nabla \bullet \left(\overline{\mathbf{U}'\mathbf{U}'} \right)_4 = \nabla \bullet \left(\overline{\mathbf{U}_1'\mathbf{U}_3'} \right) + \nabla \bullet \left(\overline{\mathbf{U}_2'\mathbf{U}_2'} \right) + \nabla \bullet \left(\overline{\mathbf{U}_3'\mathbf{U}_1'} \right)$$

Time scale of Langmuir Circulations is two orders slower than wave period

$$\partial_T \sim \alpha^2 \partial_t$$

$$\partial_T \overline{\mathbf{U}}_2 + \nabla \bullet \left(\overline{\mathbf{U}}_2 \overline{\mathbf{U}}_2 \right) + \nabla \bullet \left(\overline{\mathbf{U}'\mathbf{U}'} \right)_4 = -\nabla \overline{p}_4$$

$$\nabla \bullet \left(\overline{\mathbf{U}'\mathbf{U}'} \right)_4 = \nabla \bullet \left(\overline{\mathbf{U}'_1\mathbf{U}'_3} \right) + \nabla \bullet \left(\overline{\mathbf{U}'_2\mathbf{U}'_2} \right) + \nabla \bullet \left(\overline{\mathbf{U}'_3\mathbf{U}'_1} \right)$$

$$\partial_T \overline{\mathbf{U}}_2 + \left(\nabla \times \overline{\mathbf{U}}_2 \right) \times \overline{\mathbf{U}}_2 + \left[\overline{(\nabla \times \mathbf{U}') \times \mathbf{U}'} \right]_4 = -\nabla \left(\overline{p + |\mathbf{U}|^2 / 2} \right)_4$$

$$\left[\overline{(\nabla \times \mathbf{U}') \times \mathbf{U}'} \right]_4 = \overline{\left(\nabla \times \mathbf{U}'_1 \right) \times \mathbf{U}'_3} + \overline{\left(\nabla \times \mathbf{U}'_2 \right) \times \mathbf{U}'_2} + \overline{\left(\nabla \times \mathbf{U}'_3 \right) \times \mathbf{U}'_1} = -\partial_t \overline{\left(\nabla \times \mathbf{U}'_3 \right) \times \mathbf{X}'_1}$$

3D Eulerian framework (Craik & Leibovich, 1976)

$$\partial_t \mathbf{U}_1' = -\nabla p_1' \qquad\qquad \overline{\mathbf{U}}_1 = 0$$

$$\partial_t \mathbf{U}_2' + \mathbf{U}_1' \bullet \nabla \mathbf{U}_1' = -\nabla p_1'$$

$$\partial_t \mathbf{U}_3' + \mathbf{U}_1' \bullet \nabla \left(\overline{\mathbf{U}}_2 + \mathbf{U}_2' \right) + \left(\overline{\mathbf{U}}_2 + \mathbf{U}_2' \right) \bullet \nabla \mathbf{U}_1' = -\nabla p_3'$$

$$\nabla \times \mathbf{U}_1' = 0$$

$$\nabla \times \mathbf{U}_2' = 0$$

$$\partial_t \left(\nabla \times \mathbf{U}_3' \right) = -\nabla \times \left[\mathbf{U}_1' \bullet \nabla \left(\overline{\mathbf{U}}_2 + \mathbf{U}_2' \right) + \left(\overline{\mathbf{U}}_2 + \mathbf{U}_2' \right) \bullet \nabla \mathbf{U}_1' \right]$$

Time scale of Langmuir Circulations is two orders slower than wave period $\partial_T \sim \alpha^2 \partial_t$

$$\partial_T \overline{\mathbf{U}}_2 + \nabla \cdot \left(\overline{\mathbf{U}}_2 \overline{\mathbf{U}}_2 \right) + \nabla \cdot \left(\overline{\mathbf{U}'\mathbf{U}'} \right)_4 = -\nabla \overline{p}_4$$

$$\nabla \cdot \left(\overline{\mathbf{U}'\mathbf{U}'} \right)_4 = \nabla \cdot \left(\overline{\mathbf{U}'_1\mathbf{U}'_3} \right) + \nabla \cdot \left(\overline{\mathbf{U}'_2\mathbf{U}'_2} \right) + \nabla \cdot \left(\overline{\mathbf{U}'_3\mathbf{U}'_1} \right)$$

$$\partial_T \overline{\mathbf{U}}_2 + \left(\nabla \times \overline{\mathbf{U}}_2 \right) \times \overline{\mathbf{U}}_2 + \left[\overline{(\nabla \times \mathbf{U}') \times \mathbf{U}'} \right]_4$$

$$= -\nabla \left(\overline{p + |\mathbf{U}|^2 / 2} \right)_4$$

$$\left[\overline{(\nabla \times \mathbf{U}') \times \mathbf{U}'} \right]_4$$

$$= \cancel{\overline{\left(\nabla \times \mathbf{U}'_1 \right) \times \mathbf{U}'_3}} + \cancel{\overline{\left(\nabla \times \mathbf{U}'_2 \right) \times \mathbf{U}'_2}} + \overline{\left(\nabla \times \mathbf{U}'_3 \right) \times \mathbf{U}'_1} = -\overline{\partial_t \left(\nabla \times \mathbf{U}'_3 \right) \times \mathbf{X}'_1}$$

$$\mathbf{X}_1' = \int^t \mathbf{U}_1' dt$$

$$\mathbf{U}_2^{Stokes} = \overline{\mathbf{X}_1' \cdot \nabla \mathbf{U}_1'}$$

Craik and Leibovich momentum equations

$$\left(\partial_T + \overline{\mathbf{U}}_2 \bullet \nabla\right)\overline{\mathbf{U}}_2 + \left(\nabla \times \overline{\mathbf{U}}_2\right) \times \mathbf{U}_2^{Stokes} = -\nabla \overline{\pi}_4$$

Vortex Force

3D Lagrangian framework

(Leibovich, 1980)

Vertical	**Horizontal**	**Radiation Stress** (Reynolds Stress + Form Stress)	**Vortex Force**
Integrated	Eulerian	Longuet-Higgins & Stewart (1964)	Garrett (1976)
Eulerian	Eulerian	-	**Craik & Leibovich (1976)** McWilliams et al. (2004)
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Lagrangian	Eulerian	Mellor (2003), Aiki & Greatbatch (2012)	this study

(x,y,z) Eulerian-Cartesian coordinates

$(a,b,c)=\left(\bar{x}^L,\bar{y}^L,\bar{z}^L\right)$ Lagrangian coordinates

$(u,v,w)=d(x,y,z)/dt$

$$\begin{pmatrix} du/dt \\ dv/dt \\ dw/dt \end{pmatrix} = -\begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} p = -\begin{pmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{pmatrix}^{-1} \begin{pmatrix} \partial_a \\ \partial_b \\ \partial_c \end{pmatrix} p$$

Original Expression

$$\begin{pmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{pmatrix} \begin{pmatrix} du/dt \\ dv/dt \\ dw/dt \end{pmatrix} = -\begin{pmatrix} \partial_a \\ \partial_b \\ \partial_c \end{pmatrix} p$$

Transformed Expression

(x, y, z) Eulerian-Cartesian coordinates

$(a, b, c) = \left(\overline{x}^L, \overline{y}^L, \overline{z}^L\right)$ Lagrangian coordinates

Chain Rule

$$\begin{pmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} = \begin{pmatrix} \partial_a \\ \partial_b \\ \partial_c \end{pmatrix}$$

$$\left(x,y,z\right) \quad \text{Eulerian-Cartesian coordinates}$$

$$\left(a,b,c\right)=\left(\overline{x}^{L},\overline{y}^{L},\overline{z}^{L}\right) \quad \text{Lagrangian coordinates}$$

$$\left(x',y',z'\right)=\left(x,y,z\right)-\left(a,b,c\right)$$

Original Expression

$$\frac{d}{dt}\begin{pmatrix}\overline{u}^{L}\\ \overline{v}^{L}\\ \overline{w}^{L}\end{pmatrix}=-\overline{\begin{pmatrix}x_a & y_a & z_a\\ x_b & y_b & z_b\\ x_c & y_c & z_c\end{pmatrix}^{-1}\begin{pmatrix}\partial_a\\ \partial_b\\ \partial_c\end{pmatrix}p}^{L}$$

Transformed Expression

$$\frac{d}{dt}\underbrace{\left[\begin{pmatrix}\overline{u}^{L}\\ \overline{v}^{L}\\ \overline{w}^{L}\end{pmatrix}-\underset{\text{Pseudomomentum}}{\overline{\begin{pmatrix}-x'_a & -y'_a & -z'_a\\ -x'_b & -y'_b & -z'_b\\ -x'_c & -y'_c & -z'_c\end{pmatrix}\begin{pmatrix}u'\\ v'\\ w'\end{pmatrix}}^{L}}\right]}_{\text{Eulerian mean velocity}}+\underset{\substack{\downarrow\\ \text{Vortex Force}}}{\ldots\ldots}=-\begin{pmatrix}\partial_a\\ \partial_b\\ \partial_c\end{pmatrix}\overline{p}^{L}$$

$$\mathbf{X}_1' = \int^t \mathbf{U}_1' dt$$

$$\mathbf{U}_2^{Stokes} = \overline{\mathbf{X}_1' \cdot \nabla \mathbf{U}_1'}$$

Craik and Leibovich momentum equations

$$\left(\partial_T + \overline{\mathbf{U}}_2 \bullet \nabla\right)\overline{\mathbf{U}}_2 + \left(\nabla \times \overline{\mathbf{U}}_2\right) \times \mathbf{U}_2^{Stokes} = -\nabla \overline{\boldsymbol{\pi}}_4$$

Vortex Force

Vertically Lagrangian Horizontally Eulerian framework

(this study)

Vertical	Horizontal	Radiation Stress (Reynolds Stress + Form Stress)	Vortex Force
Integrated	Eulerian	Longuet-Higgins & Stewart (1964)	Garrett (1976)
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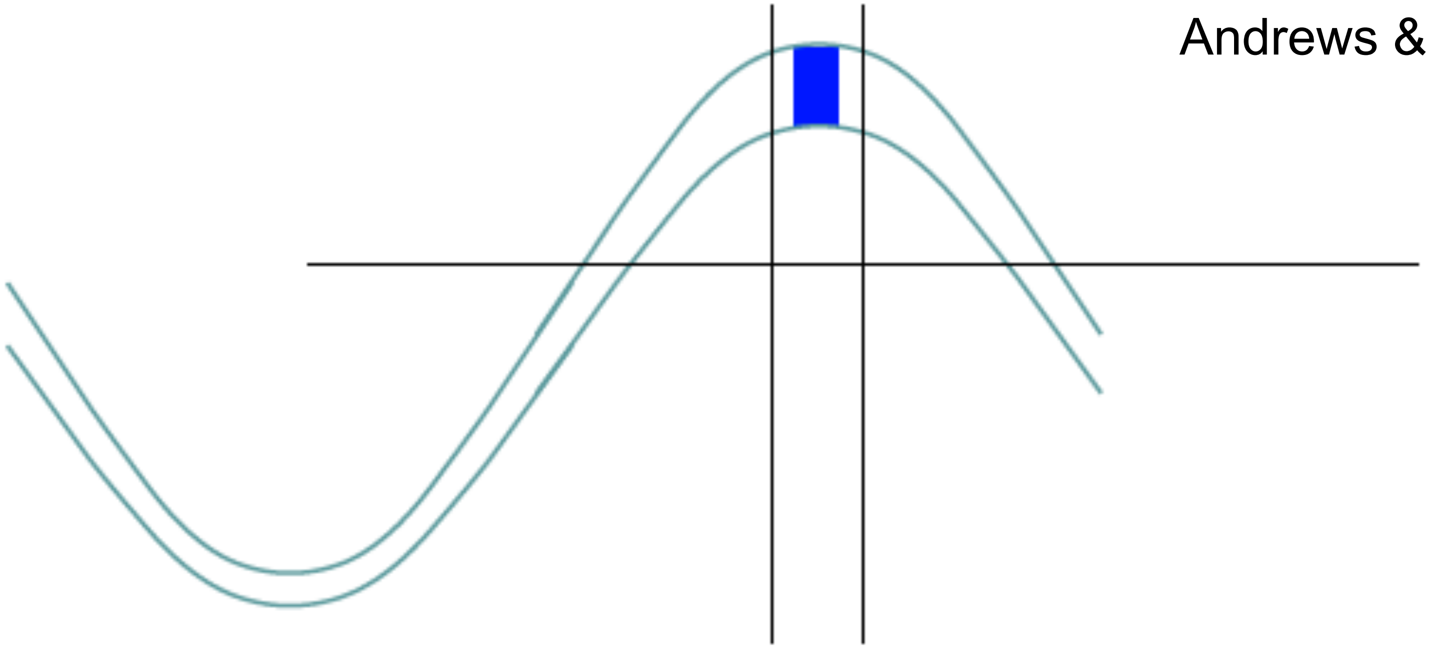
Generalized Lagrangian

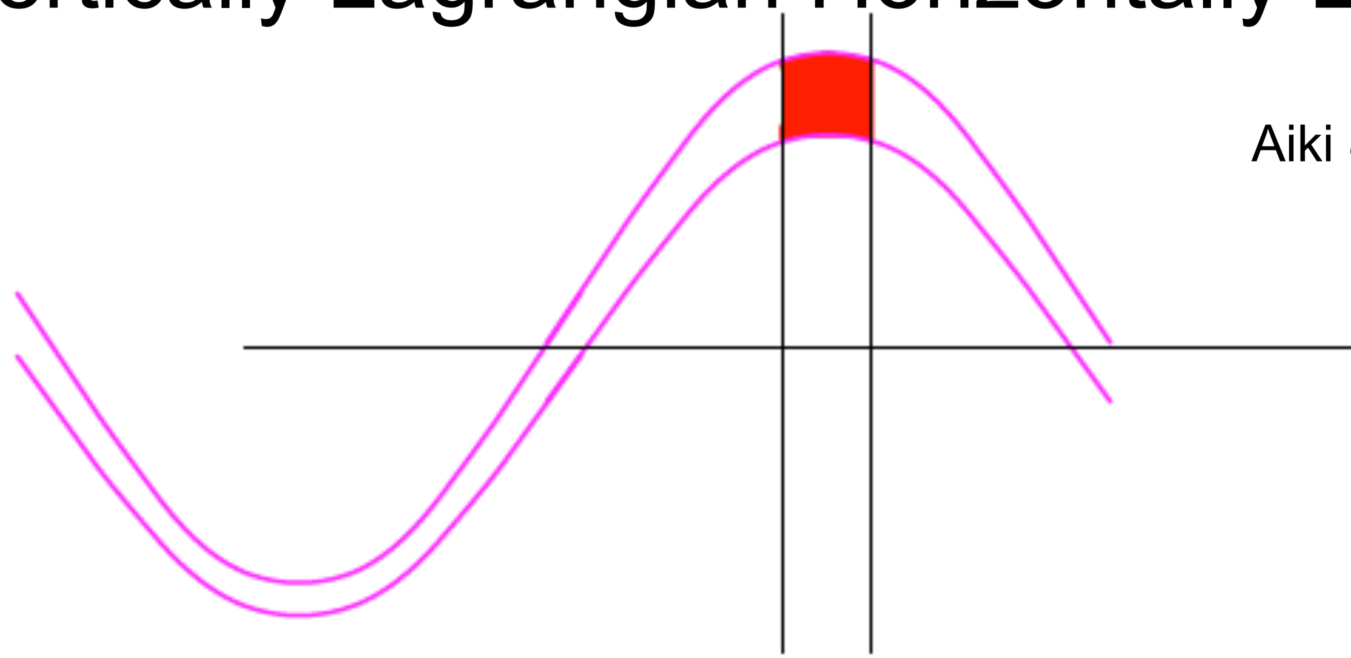
Andrews & McIntyre (1978, JFM)



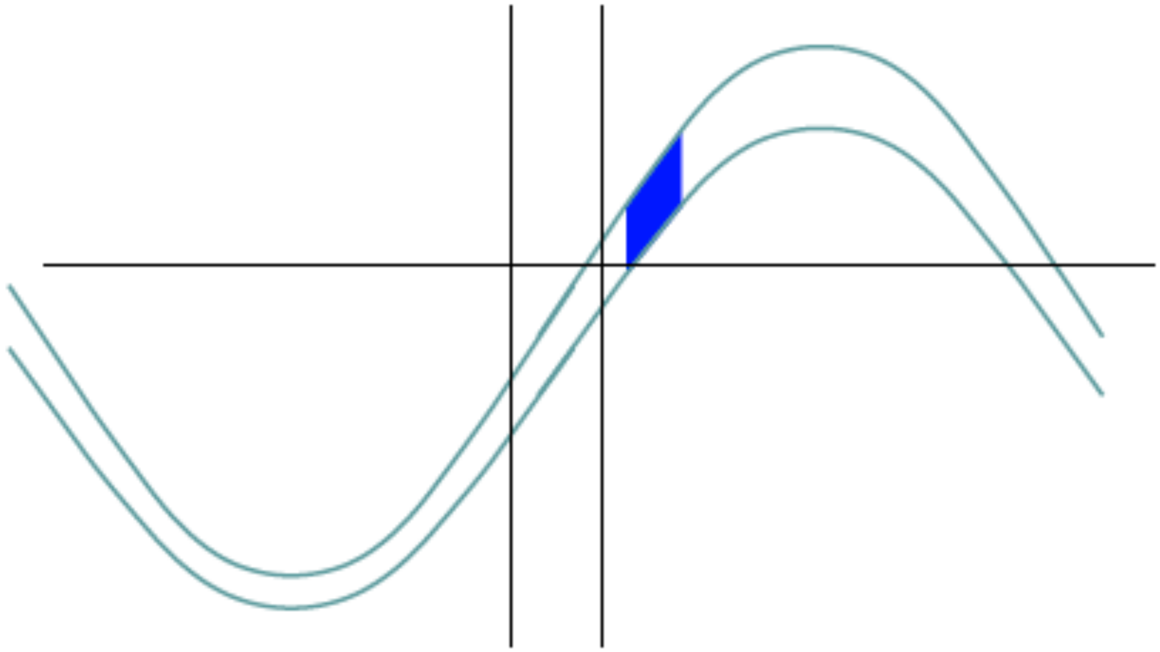
Vertically Lagrangian Horizontally Eulerian

Mellor (2003, JPO)

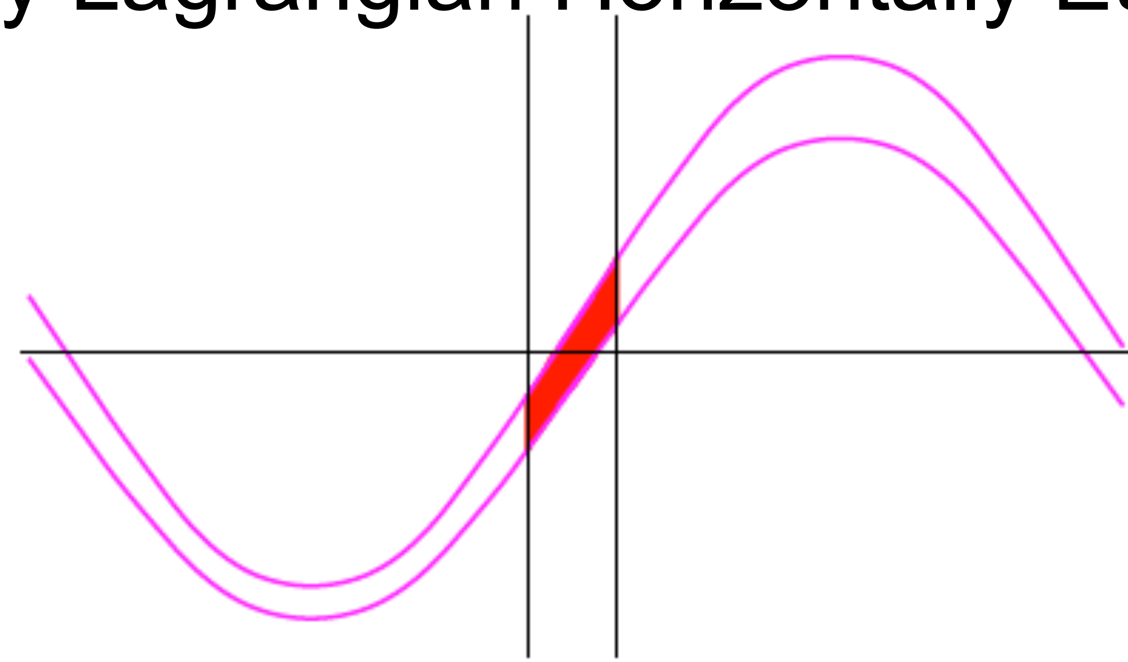
Aiki & Greatbatch (2012, JPO)



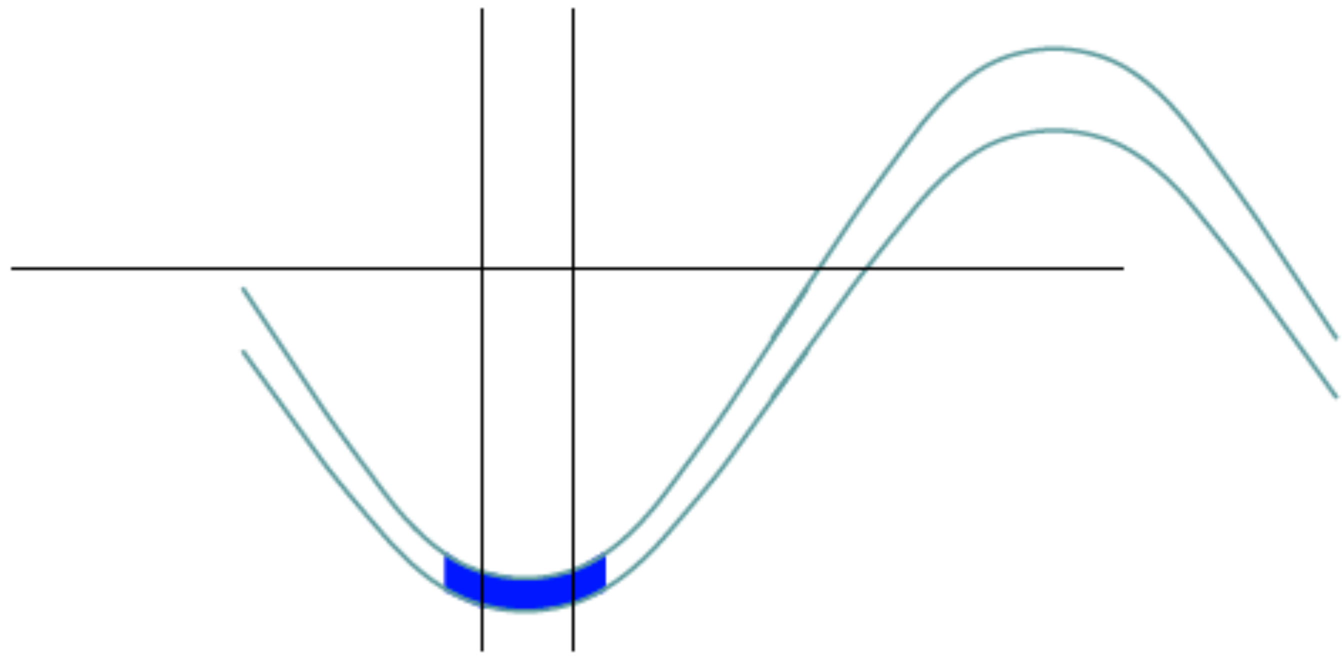
Generalized Lagrangian



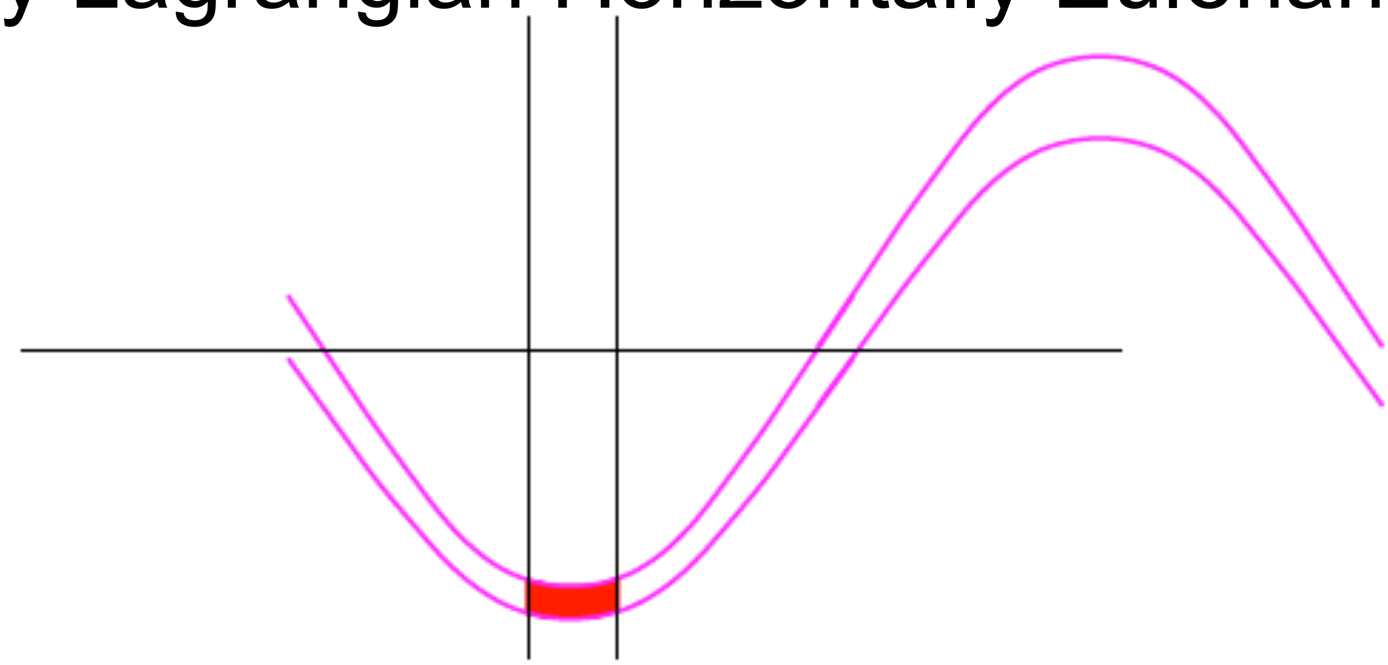
Vertically Lagrangian Horizontally Eulerian



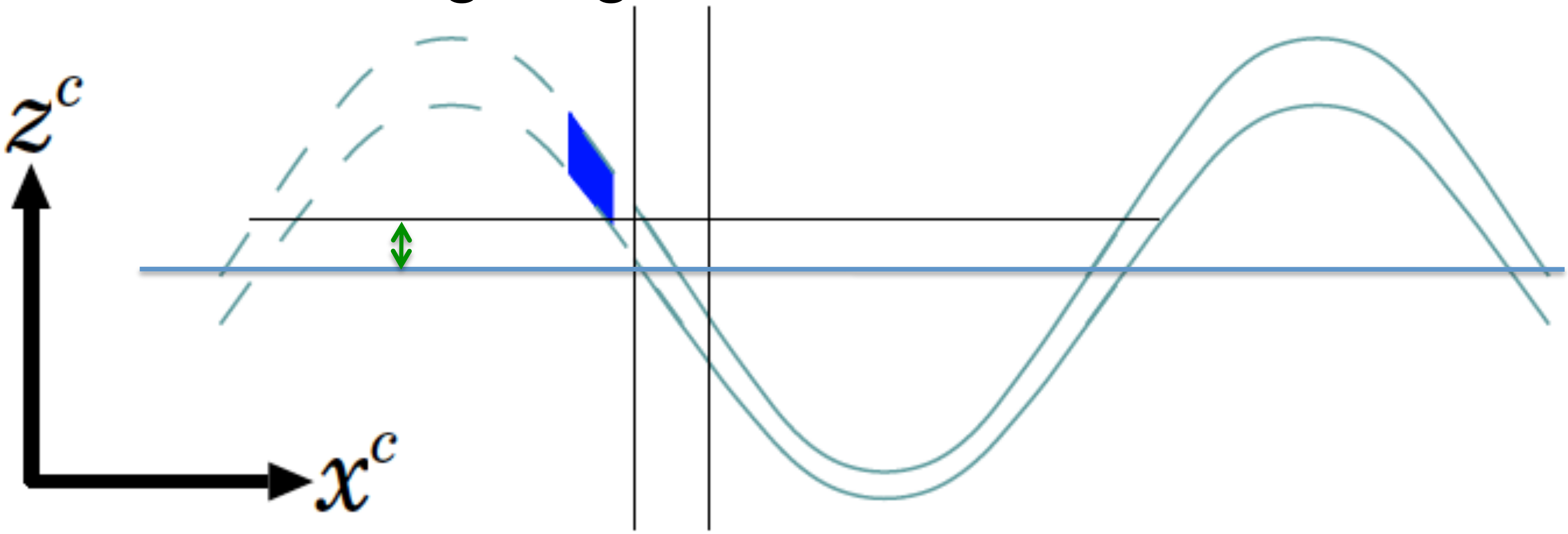
Generalized Lagrangian



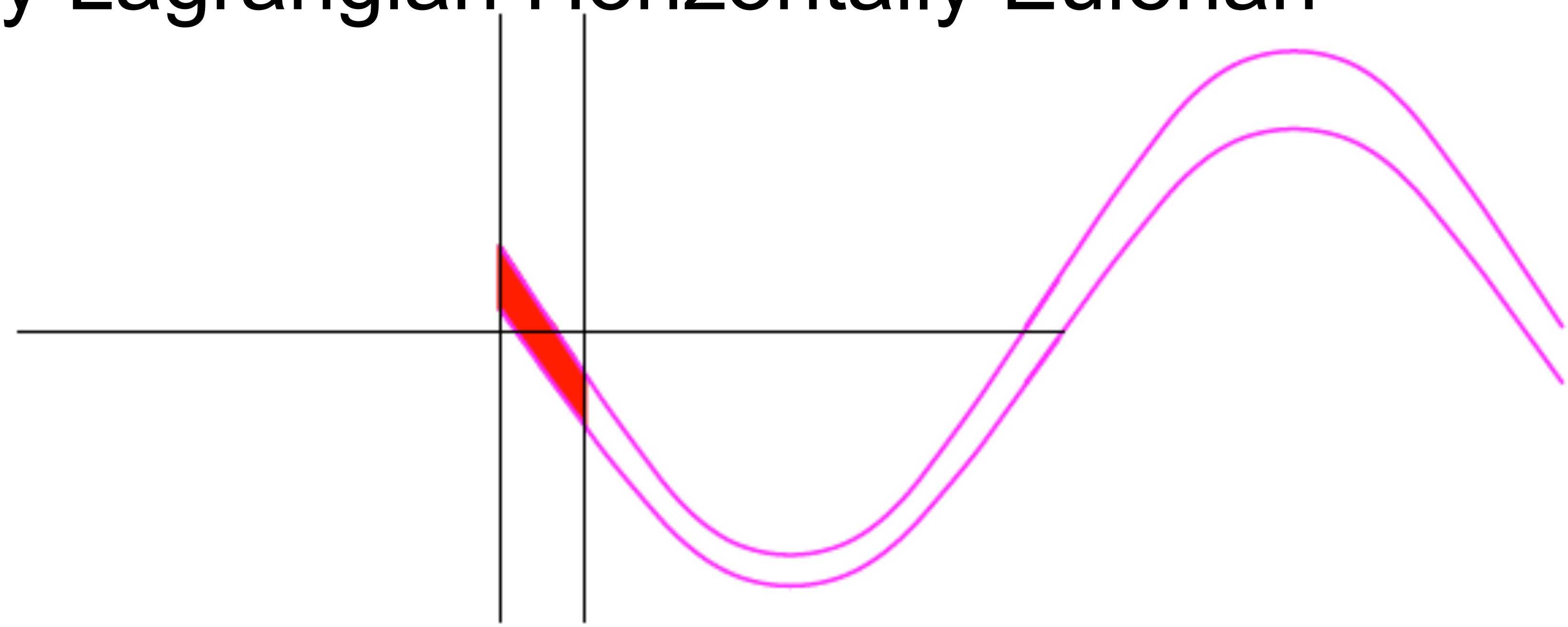
Vertically Lagrangian Horizontally Eulerian



Generalized Lagrangian

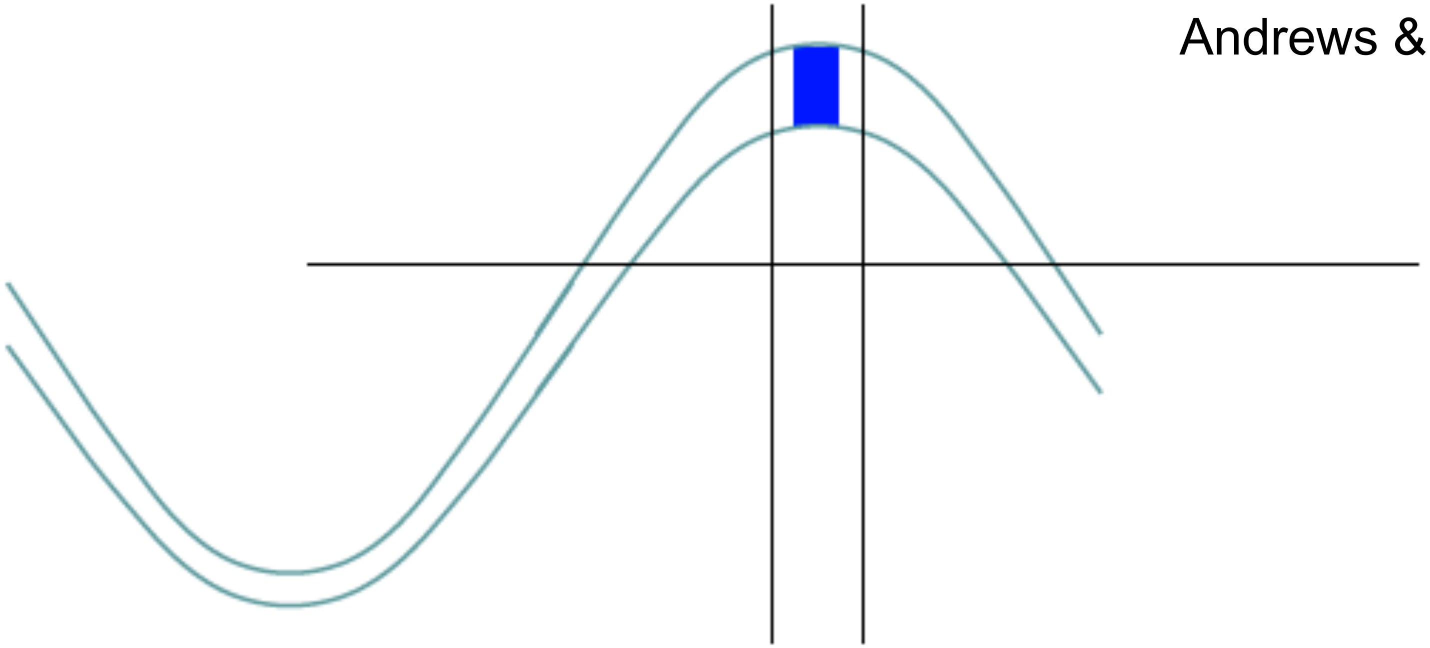


Vertically Lagrangian Horizontally Eulerian



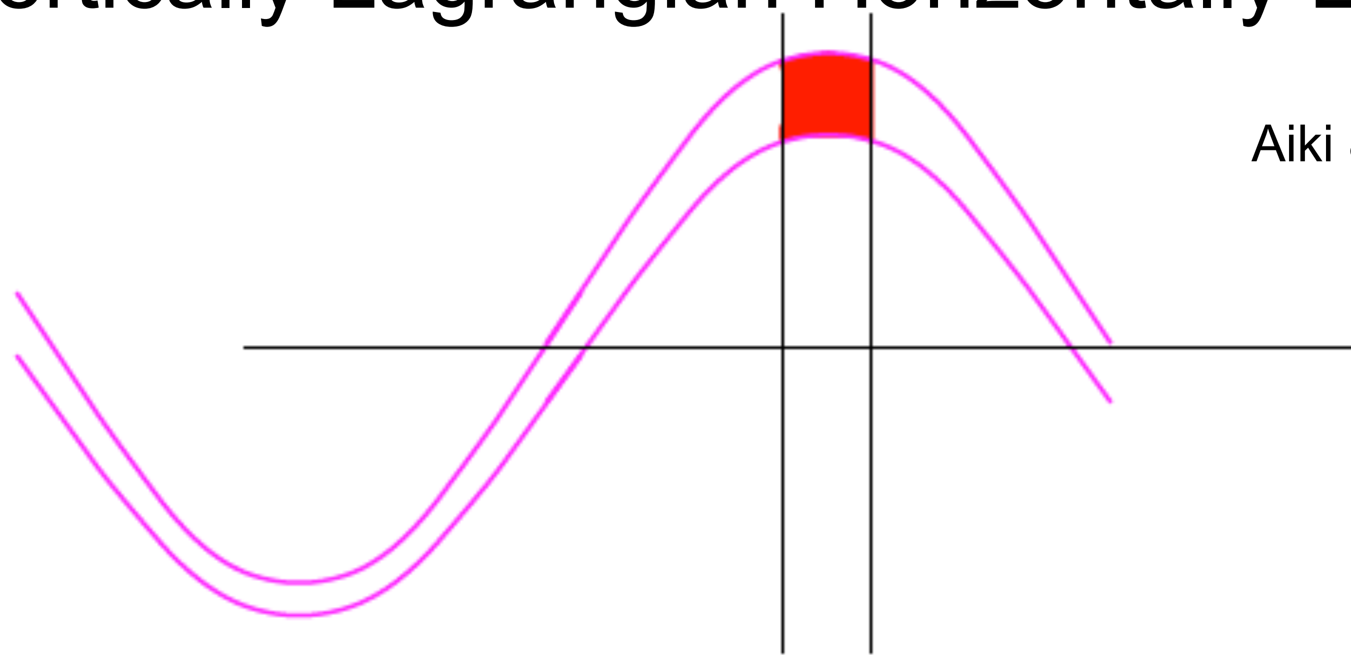
Generalized Lagrangian

Andrews & McIntyre (1978, JFM)



Vertically Lagrangian Horizontally Eulerian

Mellor (2003, JPO)
Aiki & Greatbatch (2012, JPO)



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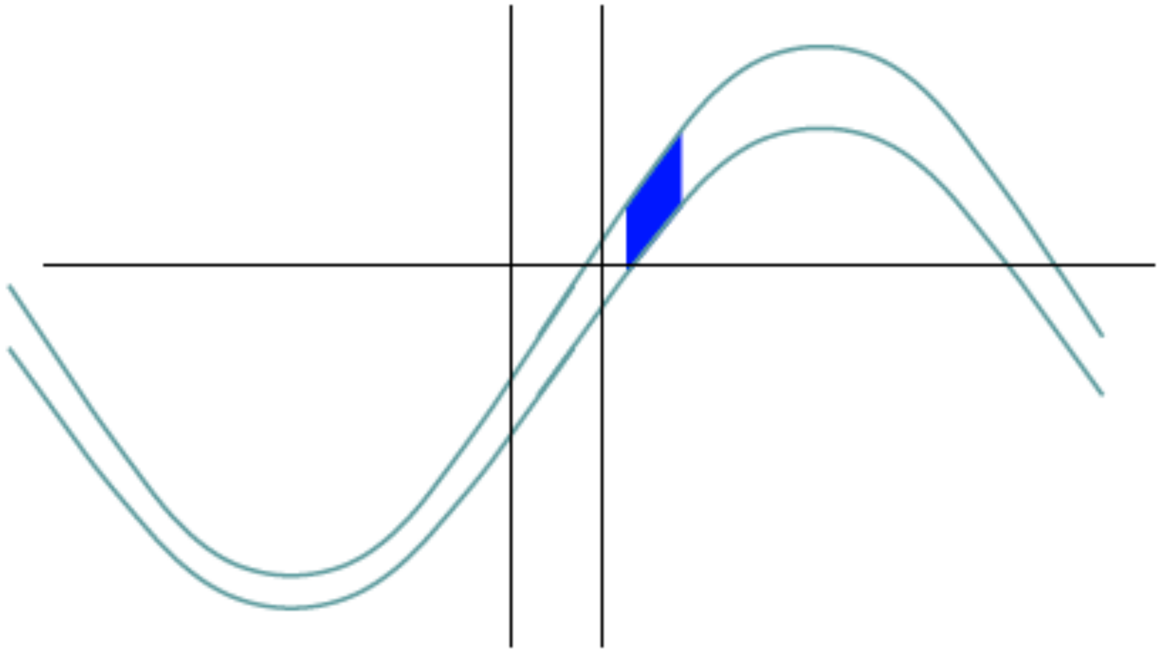

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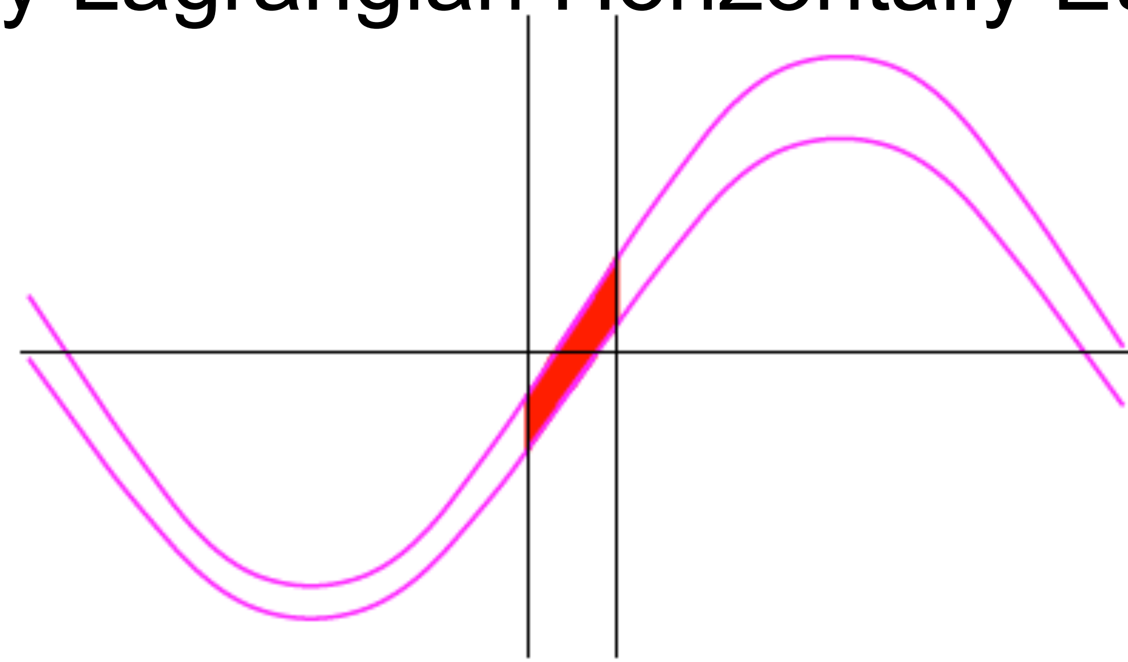

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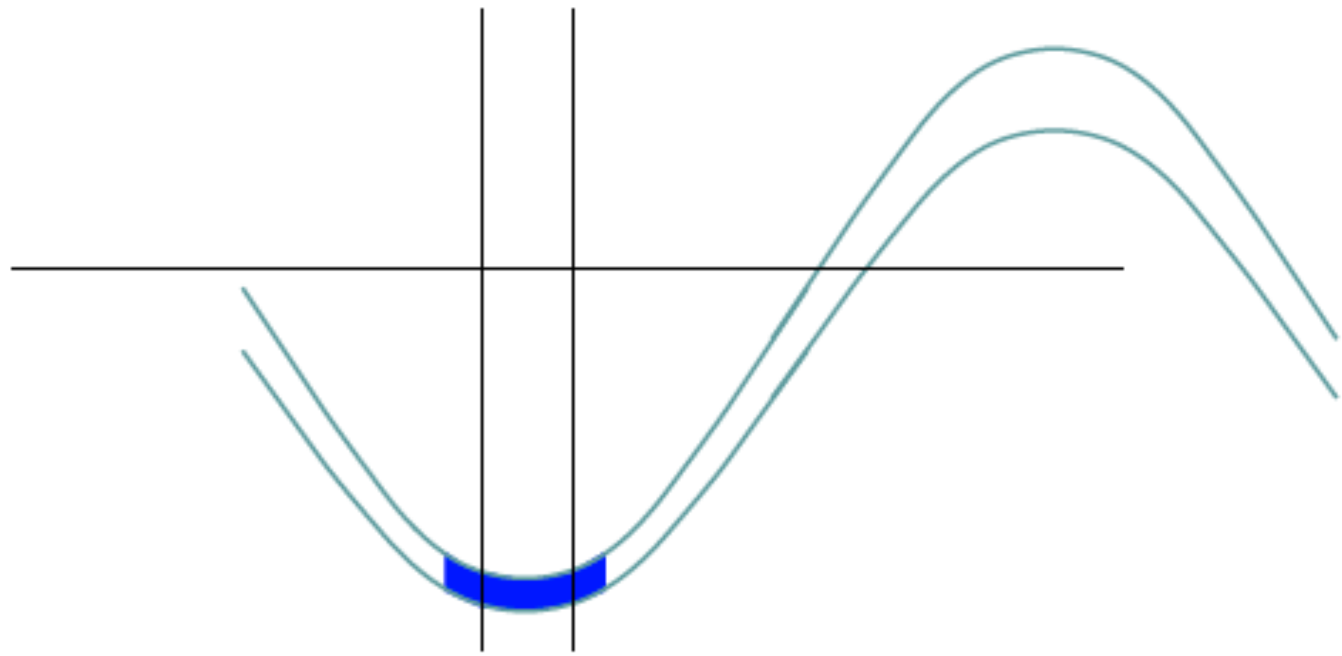

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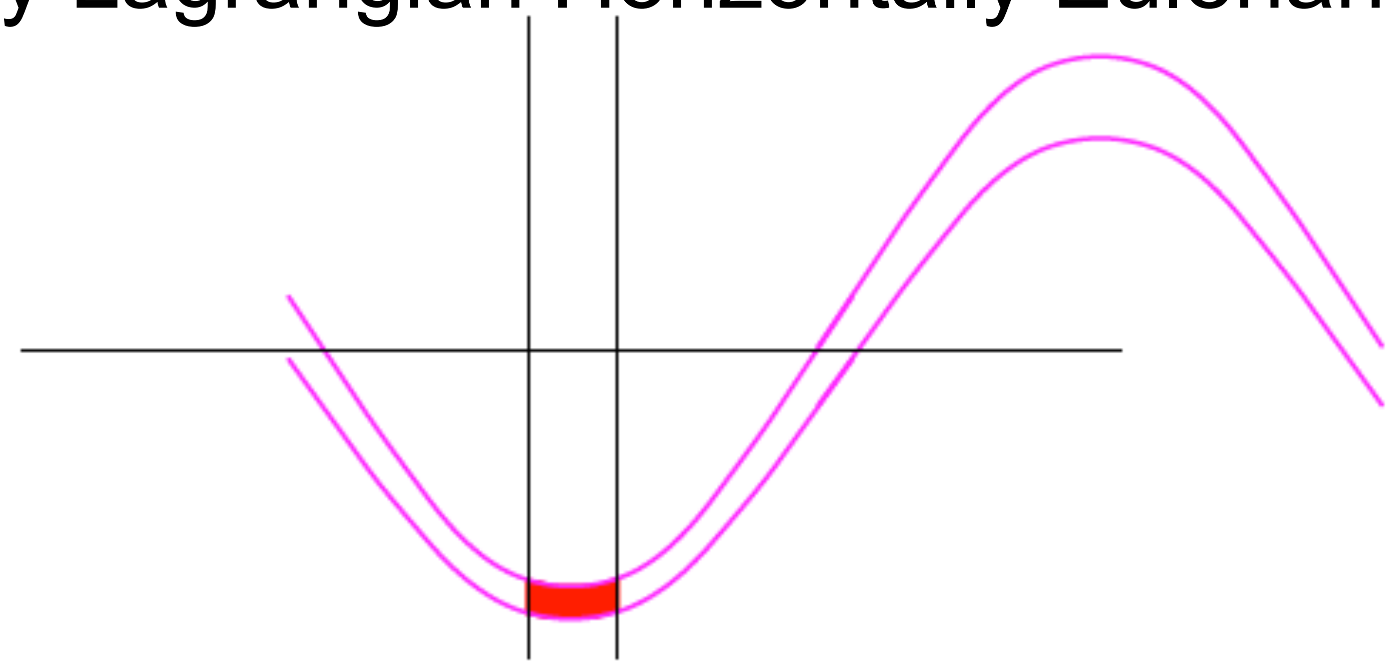
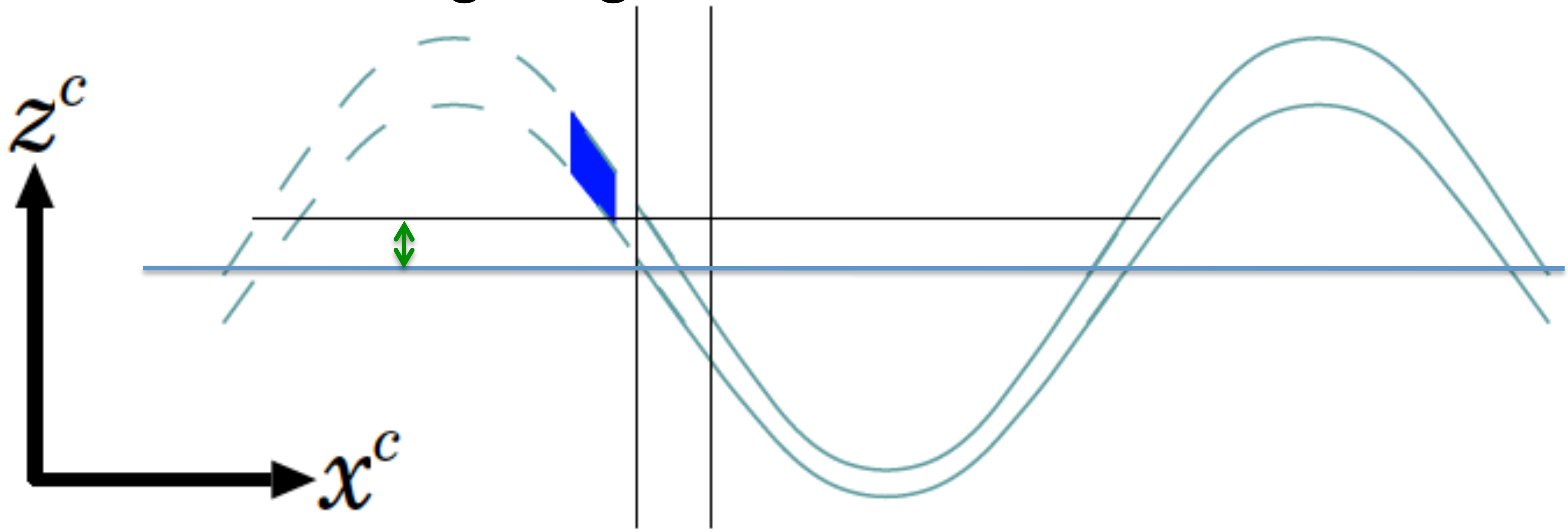
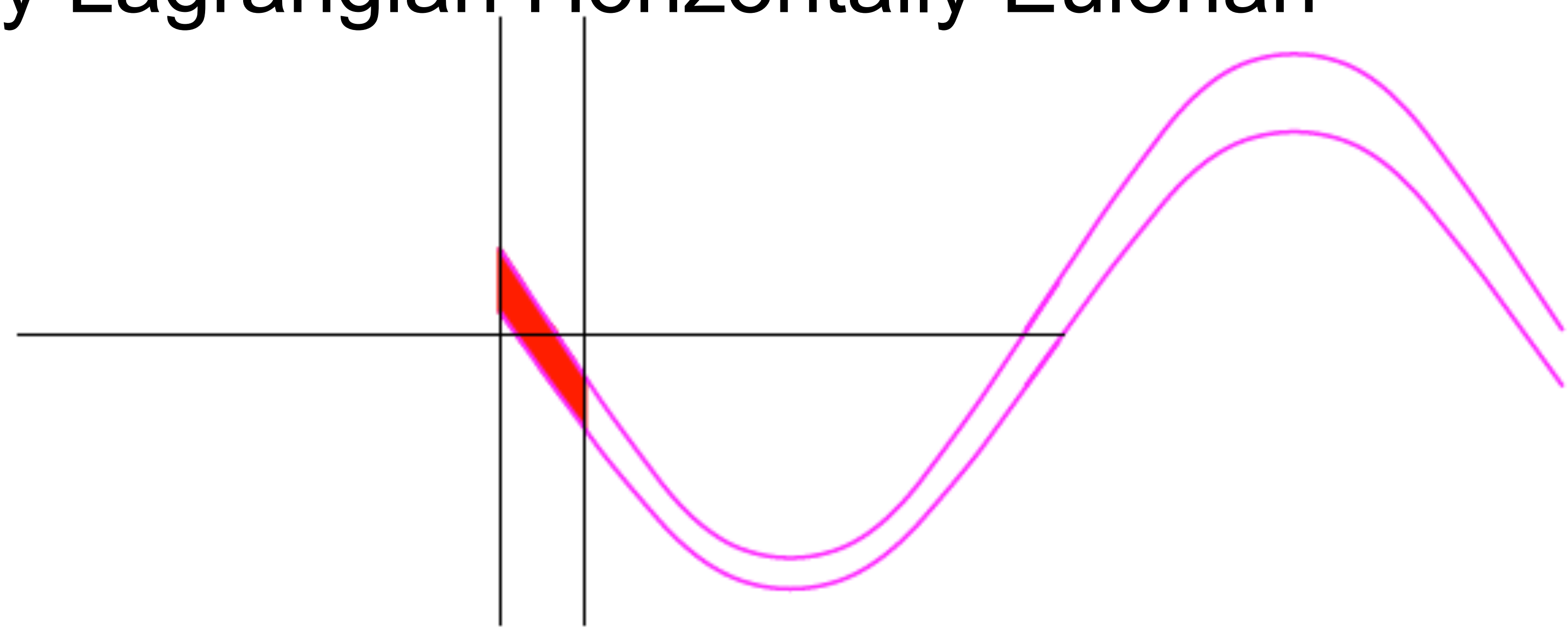


Generalized Lagrangian



Vertically Lagrangian Horizontally Eulerian



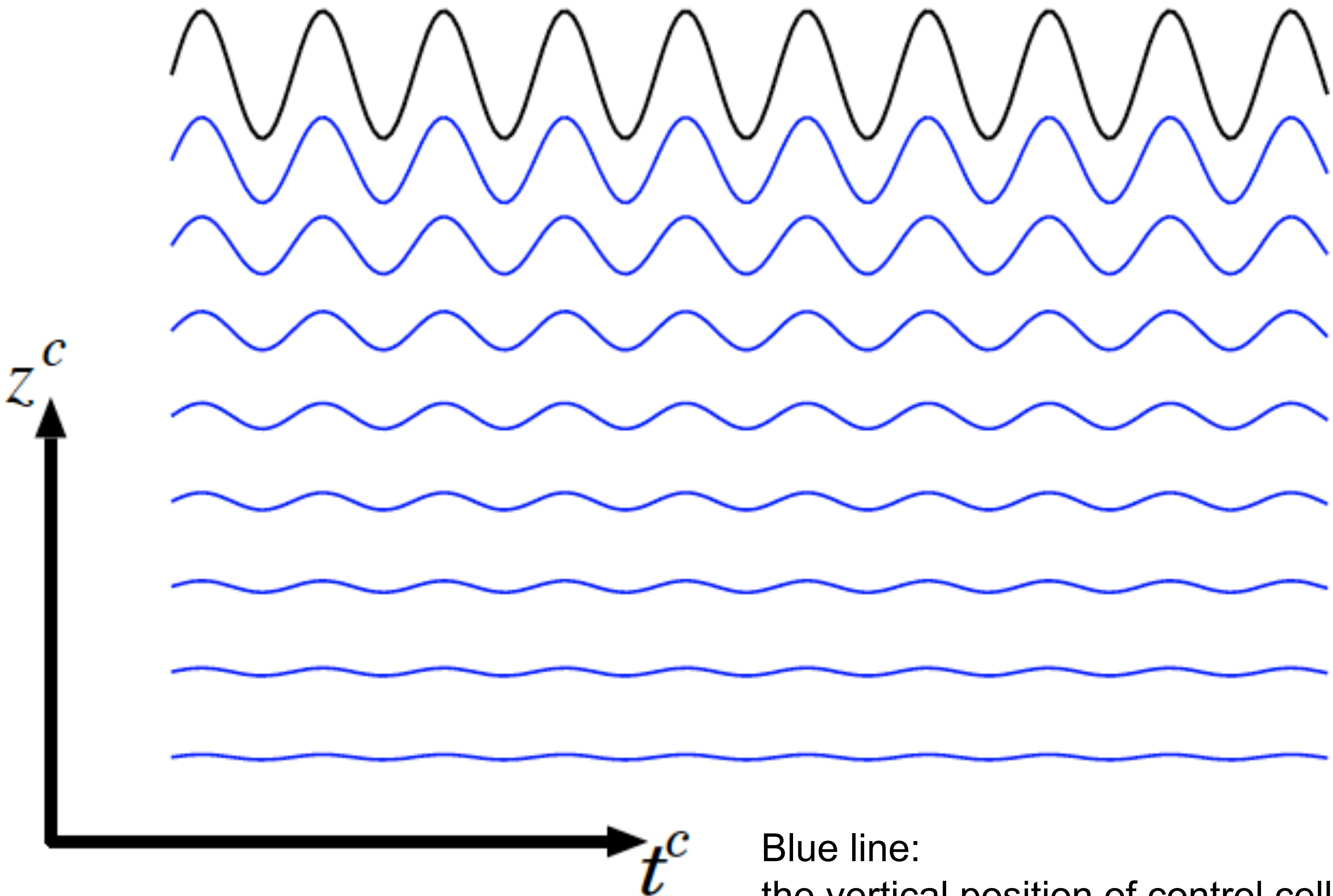
Equations in VLHE coordinates

$$(z_z^c)_t + \nabla \cdot (z_z^c \mathbf{V}) + (z_z^c w^*)_z = 0,$$

$$\begin{aligned}
(z_z^c \mathbf{V})_t + \nabla \cdot (z_z^c \mathbf{V}\mathbf{V}) + (z_z^c w^* \mathbf{V})_z &= -z_z^c \nabla (p + \eta) + p_z \nabla z^c, \\
(z_z^c w)_t + \nabla \cdot (z_z^c \mathbf{V} w) + (z_z^c w^* w)_z &= -p_z.
\end{aligned}$$

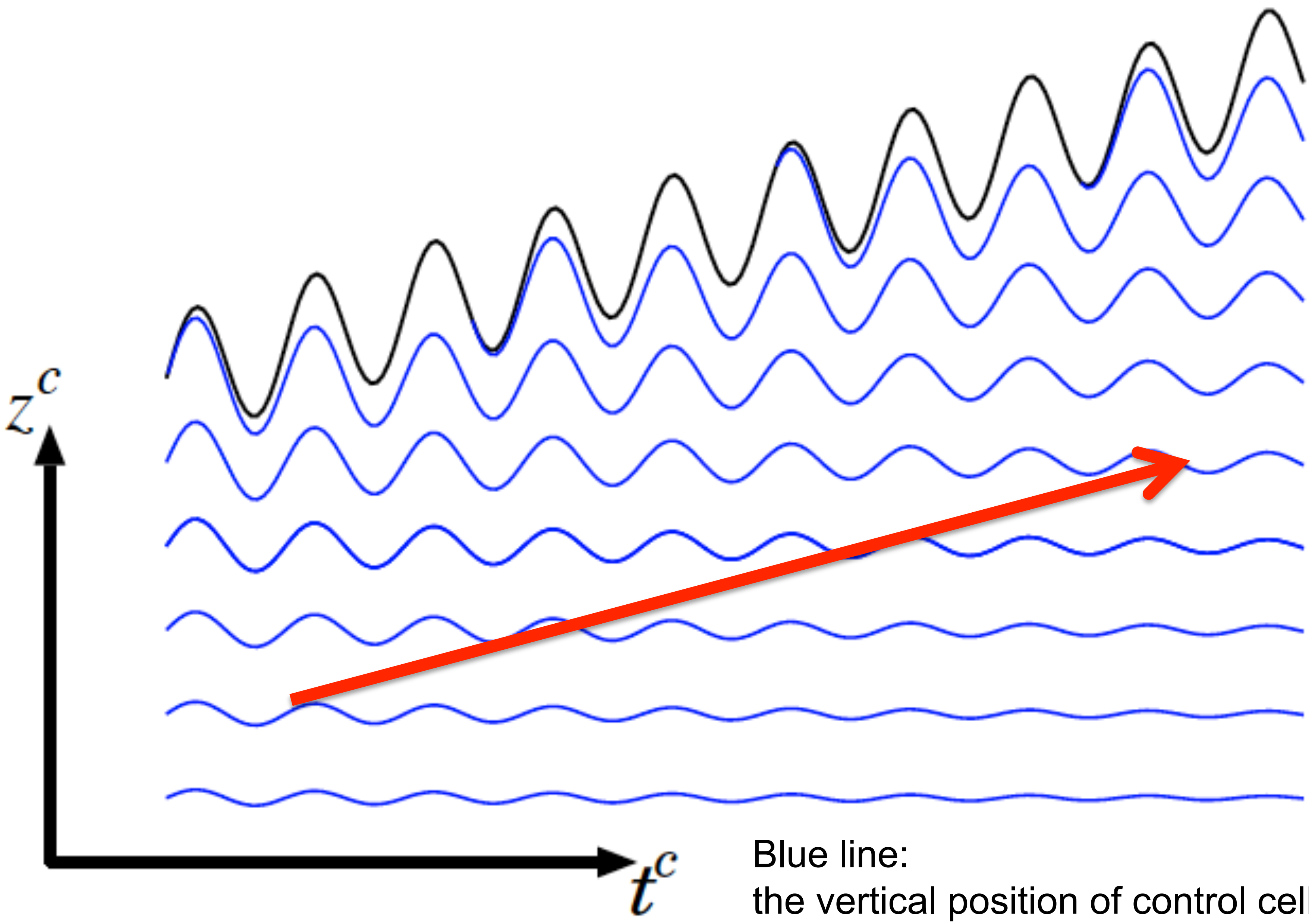
Aiki & Greatbatch (2012, JPO)

No background vertical flow



Blue line:
the vertical position of control cell

With background vertical flow（e.g. horizontal divergence of Langmuir Circulations）



Blue line:
the vertical position of control cell
Red Arrow:
the vertical position of water particle

Equations in VLHE coordinates

$$(z_z^c)_t + \nabla \cdot (z_z^c \mathbf{V}) + (z_z^c w^*)_z = 0,$$

$$\begin{aligned}
(z_z^c \mathbf{V})_t + \nabla \cdot (z_z^c \mathbf{V}\mathbf{V}) + (z_z^c w^* \mathbf{V})_z &= -z_z^c \nabla (p + \eta) + p_z \nabla z^c, \\
(z_z^c w)_t + \nabla \cdot (z_z^c \mathbf{V} w) + (z_z^c w^* w)_z &= -p_z.
\end{aligned}$$

$$\widehat{A} \equiv \overline{z_z^c A}$$

Thickness Weighted Mean operator

Thickness Weighted Mean（TWM）Equations

$$
\begin{aligned}
\nabla \cdot \widehat{\mathbf{V}} + \widehat{w^*}_z &= 0, \\
\widehat{\mathbf{V}}_t + \nabla \cdot (\widehat{\mathbf{V}}\widehat{\mathbf{V}}) + (\widehat{w^*\mathbf{V}})_z + \mathcal{RS}^{\mathbf{V}} &= -\nabla(\overline{p} + \overline{\eta}) + \mathfrak{FS}^{\mathbf{V}}, \\
\widehat{w}_t + \nabla \cdot (\widehat{\mathbf{V}}\widehat{w}) + (\widehat{w^*\widehat{w}})_z + \mathcal{RS}^{w} &= -\overline{p}_z,
\end{aligned}
$$

$(\widehat{\mathbf{V}}, \widehat{w^*})$ Total Transport Velocity

Aiki & Greatbatch (2012, JPO)

Thickness Weighted Mean（TWM）Equations

$$\nabla \cdot \widehat{\mathbf{V}} + \widehat{w^*}_z = 0,$$

$$\widehat{\mathbf{V}}_t + \nabla \cdot (\widehat{\mathbf{V}}\widehat{\mathbf{V}}) + (\widehat{w^*\mathbf{V}})_z + \mathcal{RS}^{\mathbf{V}} = -\nabla(\overline{p} + \overline{\eta}) + \mathfrak{FS}^{\mathbf{V}},$$

$$\widehat{w}_t + \nabla \cdot (\widehat{\mathbf{V}}\widehat{w}) + (\widehat{w^*\widehat{w}})_z + \mathcal{RS}^w = -\overline{p}_z,$$

Reynolds Stress

$$RS^A \equiv \nabla \bullet \left(\overline{z_z^c \mathbf{V}'' A''} \right)$$

Form Stress

$$FS^{\mathbf{V}} \equiv -\overline{z_z''' \nabla\left(p''' + \eta'''\right)} + \overline{p_z''' \nabla z'''}$$

$$= -\left[\overline{z''' \nabla\left(p''' + \eta'''\right)} \right]_z + \nabla\left(\overline{p_z''' z'''} \right)$$

(x, y, z) Eulerian-Cartesian coordinates

$(a, b, c) = \left(\overline{x}^L, \overline{y}^L, \overline{z}^L\right)$ Lagrangian coordinates

$(x', y', z') = (x, y, z) - (a, b, c)$

Original Expression

$$\frac{d}{dt}\begin{pmatrix} \overline{u}^L \\ \overline{v}^L \\ \overline{w}^L \end{pmatrix} = -\overline{\begin{pmatrix} x_a & y_a & z_a \\ x_b & y_b & z_b \\ x_c & y_c & z_c \end{pmatrix}^{-1} \begin{pmatrix} \partial_a \\ \partial_b \\ \partial_c \end{pmatrix} p}^L$$

Transformed Expression

$$\frac{d}{dt}\underbrace{\left[\begin{pmatrix} \overline{u}^L \\ \overline{v}^L \\ \overline{w}^L \end{pmatrix} - \underbrace{\overline{\begin{pmatrix} -x'_a & -y'_a & -z'_a \\ -x'_b & -y'_b & -z'_b \\ -x'_c & -y'_c & -z'_c \end{pmatrix}\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix}}^L}_{\text{Pseudomomentum}}\right]}_{\text{Eulerian mean velocity}} + \underbrace{\ldots\ldots}_{\text{Vortex Force}} = -\begin{pmatrix} \partial_a \\ \partial_b \\ \partial_c \end{pmatrix} \overline{p}^L$$

Form Stress Term

$$\begin{aligned}
\mathfrak{F}\mathfrak{S}^{\mathbf{V}} &= -\overline{z_z'''\nabla(p''' + \eta''')} + \overline{(\nabla z''')p_z'''} \\
&= -\overline{z_z'''\nabla(p + \eta)} + \overline{(\nabla z''')p_z} \\
&= \overline{z_z'''(\mathfrak{D}_t\mathbf{V} + (\mathfrak{D}_t w)\nabla z''')} - \overline{\nabla z'''(1 + z_z''')(\mathfrak{D}_t w)} \\
&= \overline{z_z'''(\mathfrak{D}_t\mathbf{V})} - \overline{\nabla z'''(\mathfrak{D}_t w)} \\
&= \overline{z_z'''(\mathfrak{D}_t\mathbf{V})'''} - \overline{\nabla z'''(\mathfrak{D}_t w)'''},
\end{aligned}$$

Prognostic variable in the **original** equation system

$(\widehat{\mathbf{V}}, \widehat{w^*})$ Total Transport Velocity

Prognostic variable in the **transformed** equation system

$$\begin{pmatrix} \overline{u}_2 + \overline{z'''_{1x} w'''_1} \\ \overline{v}_2 + \overline{z'''_{1y} w'''_1} \\ \overline{w}_2 + \overline{z'''_{1z} w'''_1} \end{pmatrix} = \underbrace{\begin{pmatrix} \widehat{u}_2 \\ \widehat{v}_2 \\ \widehat{w^*_2} \end{pmatrix}}_{\text{total transport velocity}} - \underbrace{\overline{\begin{pmatrix} +z'''_{1z} & 0 & -z'''_{1x} \\ 0 & +z'''_{1z} & -z'''_{1y} \\ -z'''_{1x} & -z'''_{1y} & -z'''_{1z} \end{pmatrix} \begin{pmatrix} u'''_1 \\ v'''_1 \\ w'''_1 \end{pmatrix}}}_{\text{pseudomomentum}},$$

Prognostic variable in the **original** equation system

$$(\widehat{\mathbf{V}}, \widehat{w^*})$$

Total Transport Velocity
= Lagrangian mean velocity

Prognostic variable in the **transformed** equation system

$$\begin{pmatrix} \overline{u}_2 + \overline{z'''_{1x} w'''_1} \\ \overline{v}_2 + \overline{z'''_{1y} w'''_1} \\ \overline{w}_2 + \overline{z'''_{1z} w'''_1} \end{pmatrix} = \underbrace{\begin{pmatrix} \widehat{u}_2 \\ \widehat{v}_2 \\ \widehat{w^*_2} \end{pmatrix}}_{\text{total transport velocity}} - \underbrace{\overline{\begin{pmatrix} +z'''_{1z} & 0 & -z'''_{1x} \\ 0 & +z'''_{1z} & -z'''_{1y} \\ -z'''_{1x} & -z'''_{1y} & -z'''_{1z} \end{pmatrix} \begin{pmatrix} u'''_1 \\ v'''_1 \\ w'''_1 \end{pmatrix}}}_{\text{pseudomomentum}},$$

Eulerian mean velocity = Lagrangian mean velocity - Stokes-drift velocity

$$\mathbf{X}_1' = \int^t \mathbf{U}_1' dt$$

$$\mathbf{U}_2^{Stokes} = \overline{\mathbf{X}_1' \cdot \nabla \mathbf{U}_1'}$$

Craik and Leibovich momentum equations

$$\left(\partial_T + \overline{\mathbf{U}}_2 \bullet \nabla\right)\overline{\mathbf{U}}_2 + \left(\nabla \times \overline{\mathbf{U}}_2\right) \times \mathbf{U}_2^{Stokes} = -\nabla \overline{\pi}_4$$

Vortex Force

Summary

Vertical	Horizontal	Radiation Stress	Vortex Force
Integrated	Eulerian	Longuet-Higgins & Stewart (1964)	Garrett (1976)
Eulerian	Eulerian	-	**Craik & Leibovich (1976)** ~~Flux-divergence form (Reynolds stress)~~ Vector-invariant form (3rd order vorticity Eq.)
Lagrangian	Lagrangian	-	**Leibovich (1980)** ~~Original expression (Form stress):~~ ~~Prognostic variable: LM velocity~~ Transformed expression Prognostic variable: EM velocity = LM velocity - psudomomentum
Lagrangian	Eulerian	Mellor (2003) Aiki & Greatbatch (2012) ➢Horizontal Reynolds stress ➢Form stress ➢Prognostic variable: Total Transport velocity	➢Horizontal Reynolds stress term is transformed as in CL76 ➢Form stress term is transformed as in Leibovich (1980) ➢Prognostic variable: EM velocity: =Total Transport velocity - pseudomomentum

Studies for the Effect of Surface Waves on Circulation

Vertical	Horizontal	Incompressibility	Surface Boundary	Viscosity Term	Spectrum Shape
Eulerian	Eulerian	◎	○	○	○
Lagrangian	Lagrangian	○	○	○	◎
Lagrangian	Eulerian	◎	◎	◎	○

Generalized Lagrangian Mean theory

(Andrews & McIntyre, 1978)

- Prognostic variable in wave-averaged momentum Eqs.

  Original expression: LM velocity

  Transformed expression: LM velocity - pseudomomentum

  (Lamb, 1932, page 13; Lagrange, 1788, page 441-447)

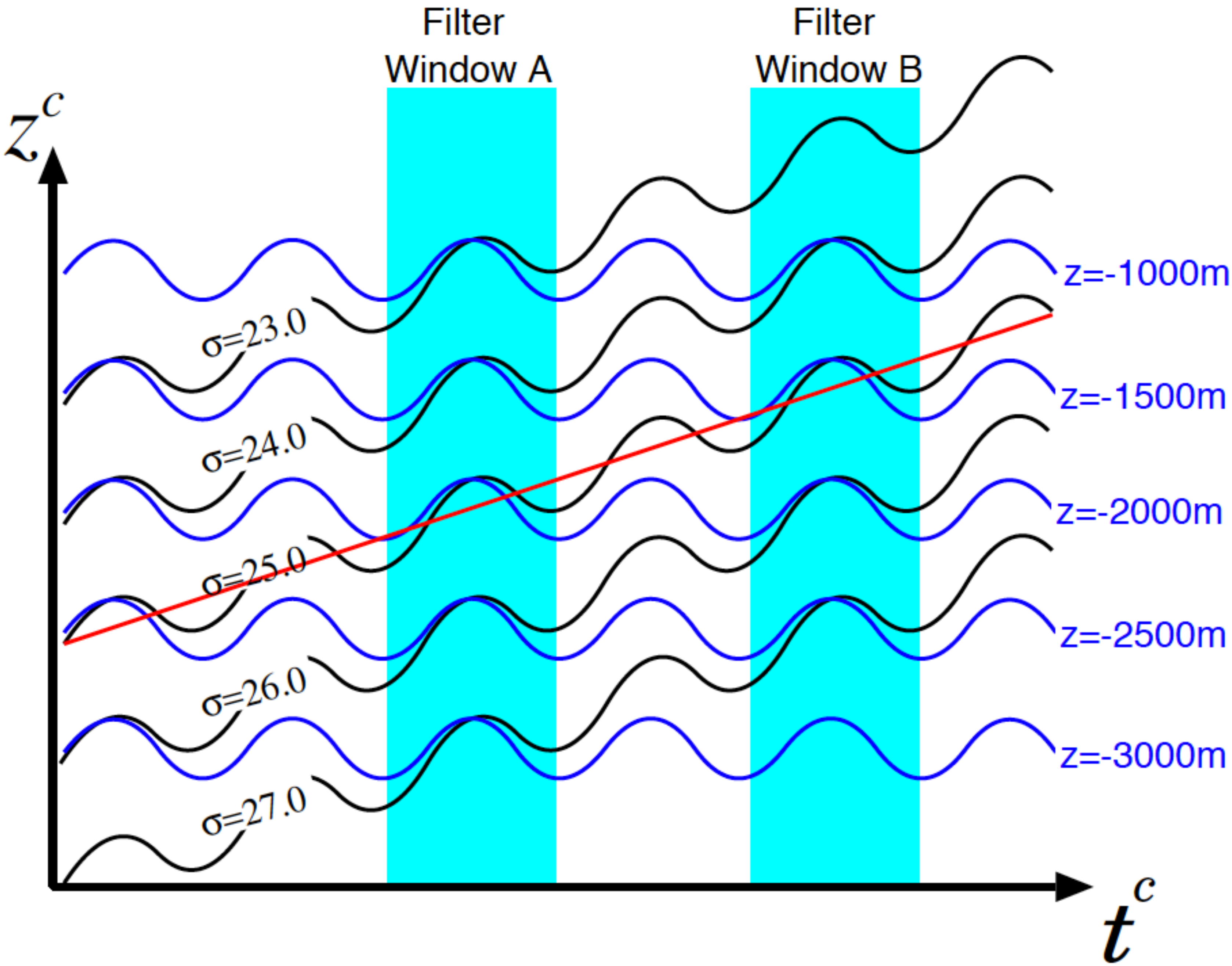
- Hybrid Lagrangian-Eulerian coordinates

  (Iwasaki, 2001; Jacobson and Aiki, 2006)

- Lagrangian pseudomomentum

black line: surface of fixed density
blue line: surface of fixed z

Jacobson & Aiki, 2006, JPO



Stokes-drift velocity and its relatives

Stokes drift velocity

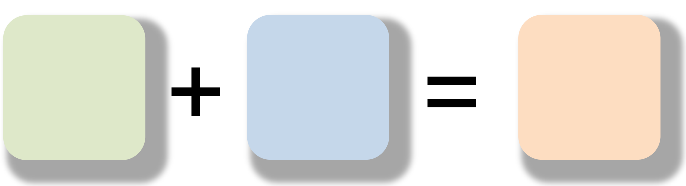
(Longuet-Higgins, 1953)

$$\overline{u'_x x' + u'_z z'}$$

Quasi-Stokes velocity

(Andrews & McIntyre, 1976;
Iwasaki, 2001;
McDougall & McIntosh, 2001)

$$\left(\overline{u'z'}\right)_z$$



Eulerian pseudomomentum

(Bretherton, 1971; Buhler, 2009)

$$-\overline{\left(u'_z - w'_x\right) z'} \,/\, 2$$

Lagrangian pseudomomentum

(Andrews & McIntyre, 1978)

$$\overline{-x'_x u' - z'_x w'}$$

TWM pseudomomentum

(Aiki & Greatbatch, under revision)

$$\overline{z'_z u' - z'_x w'}$$

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