Approximations that preserve conservation laws

KEY IDEA: Do not disturb the corresponding symmetry property.

Approximations can often be regarded as <u>constraints</u>.

First example: Constant-density flow

Exact dynamics:

$$\delta \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) \right\} = 0$$

<u>Constraint</u>:

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = \alpha_0 \qquad\qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\delta \iiint d\mathbf{a}\ E(\alpha_0, S(\mathbf{a})) = 0$$

Approximate dynamics:

$$\delta \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - \Phi(\mathbf{x}) + \lambda(\mathbf{a},\tau) \left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \alpha_0 \right) \right\} = 0$$

$$\delta \mathbf{x}: \qquad \frac{\partial^2 \mathbf{x}}{\partial \tau^2} = -\alpha_0 \nabla \lambda - \nabla \Phi$$

$$\delta \lambda: \qquad \nabla \cdot \mathbf{v} = 0$$

Second Example: Constant-density flow in a thin layer

Constraint: Fluid moves in vertical columns.

Eulerian statement:

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$$

Lagrangian statement:

$$x = x(a,b,\tau), \qquad y = y(a,b,\tau)$$

$$\frac{\partial(x,y,z)}{\partial(a,b,c)} = \alpha_0 \quad \text{becomes} \quad \frac{\partial(x,y)}{\partial(a,b)}\frac{\partial z}{\partial c} = \alpha_0$$

which integrates to

$$z = \frac{\partial(a,b)}{\partial(x,y)}\alpha_0 c \ + \ const$$

Assigning c=0 at the bottom and $c = H_0$ at the free surface z=h, we obtain:

$$z = \frac{c}{H_0}h \qquad \text{where} \qquad h = \alpha_0 H_0 \frac{\partial(a,b)}{\partial(x,y)}$$

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We build the constraints into the Lagrangian by using this relation to eliminate $z(a,b,c,\tau)$ in favor of $x(a,b,\tau)$ and $y(a,b,\tau)$.

The terms in the Lagrangian are:

$$\iint da\, db \int_0^{H_0} dc \left\{ \tfrac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \tfrac{1}{2}\left(\frac{\partial y}{\partial \tau}\right)^2 \right\} = \tfrac{1}{2} H_0 \iint da\, db \left\{ \left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 \right\}$$

$$\iint da\, db \int_0^{H_0} dc \left\{ \tfrac{1}{2}\left(\frac{\partial z}{\partial \tau}\right)^2 \right\} = \iint da\, db \int_0^{H_0} dc \left\{ \tfrac{1}{2}\left(\frac{c}{H_0}\frac{\partial h}{\partial \tau}\right)^2 \right\} = \tfrac{1}{2} H_0 \iint da\, db \left\{ \tfrac{1}{3}\left(\frac{\partial h}{\partial \tau}\right)^2 \right\}$$

$$\iint da\, db \int_0^{H_0} dc \{gz\} = \iint da\, db \int_0^{H_0} dc \left\{ g \frac{c}{H_0} h \right\} = \tfrac{1}{2} H_0 \iint da\, db \{gh\}$$

The resulting Lagrangian is

$$L[x(a,b,\tau), y(a,b,\tau)] = \tfrac{1}{2} \iint da\, db \left\{ \left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 + \tfrac{1}{3}\left(\frac{\partial h}{\partial \tau}\right)^2 - gh \right\}$$

where h is to be considered an abbreviation for $\alpha_0 H_0 \frac{\partial(a,b)}{\partial(x,y)}$

Thus Hamilton's principle takes the form:

$$\delta x: \qquad \delta\int L\, d\tau = \int d\tau \iint da\, db\left\{\frac{\partial x}{\partial \tau}\frac{\partial \delta x}{\partial \tau} + \tfrac{1}{3}\frac{\partial h}{\partial \tau}\frac{\partial \delta h}{\partial \tau} - \tfrac{1}{2} g\,\delta h\right\}$$

$$= \iint da\, db\left\{-\frac{\partial^2 x}{\partial \tau^2}\delta x - \tfrac{1}{3}\frac{\partial^2 h}{\partial \tau^2}\delta h - \tfrac{1}{2} g\,\delta h\right\}$$

Again we need an identity

$$\iint da\, db\{F\delta h\} = \iint da\, db\left\{-Fh^2\delta\left(\frac{1}{h}\right)\right\}$$

$$= \iint da\, db\left\{-Fh^2\delta\left(\frac{1}{\alpha_0 H_0}\frac{\partial(x,y)}{\partial(a,b)}\right)\right\}$$

$$= \frac{1}{\alpha_0 H_0}\iint da\, db\left\{-Fh^2\frac{\partial(\delta x,y)}{\partial(a,b)}\right\}$$

$$= \frac{1}{\alpha_0 H_0}\iint da\, db\left\{\delta x\frac{\partial(Fh^2,y)}{\partial(a,b)}\right\}$$

$$= \iint da\, db\left\{\delta x\frac{1}{h}\frac{\partial}{\partial x}\left(Fh^2\right)\right\}$$

Putting

$$F = -\frac{1}{3}\frac{\partial^2 h}{\partial \tau^2} - \frac{1}{2}g$$

we obtain

$$\delta\mathbf{x}: \qquad \frac{D\mathbf{u}}{Dt} = -g\nabla h - \frac{1}{3h}\nabla\left(h^2\frac{D^2 h}{Dt^2}\right)$$

As in the 3D case, the definition

$$h = \alpha_0 H_0 \frac{\partial(a,b)}{\partial(x,y)}$$

implies

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0$$

so the complete dynamics consists of the continuity equation and momentum equation

$$\frac{D\mathbf{u}}{Dt} = -g\nabla h - \frac{1}{3h}\nabla\left(h^2 \frac{D^2 h}{Dt^2}\right)$$

These equations were discovered by Green and Naghdi (1976) using a method based on "Cosserat surfaces."

If we completely omit the vertical kinetic energy (*very* thin layer) we obtain the Lagrangian

$$L[x(a,b,\tau),y(a,b,\tau)] = \tfrac{1}{2}\iint da\,db\left\{\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 - gh\right\}$$

for the shallow-water equations:

$$\frac{Dh}{Dt} + h\nabla \cdot \mathbf{u} = 0$$

$$\frac{D\mathbf{u}}{Dt} = -g\nabla h$$

Conservation Laws

Momentum, energy, …..potential vorticity.

The particle-relabeling symmetry is present, because the derivatives

$$\frac{\partial x}{\partial a},\ \frac{\partial x}{\partial b},\ \frac{\partial y}{\partial a},\ \frac{\partial y}{\partial b}$$

enter the Lagrangian only through

$$h = \alpha_0 H_0 \frac{\partial(a,b)}{\partial(x,y)}$$

As before

$$\delta \frac{\partial(a,b)}{\partial(x,y)} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial a}\delta a + \frac{\partial}{\partial b}\delta b = 0 \quad \Rightarrow \quad \delta a = -\frac{\partial \psi}{\partial b}, \quad \delta b = +\frac{\partial \psi}{\partial a}$$

We have

$$\delta L = \delta \ \tfrac{1}{2} \iint da\, db \left\{ \left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 + \tfrac{1}{3}\left(\frac{\partial h}{\partial \tau}\right)^2 - gh \right\}$$

$$= \iint da\, db \left\{ \frac{\partial x}{\partial \tau}\ \delta \frac{\partial x}{\partial \tau} + \frac{\partial y}{\partial \tau}\ \delta \frac{\partial y}{\partial \tau} + \tfrac{1}{3} \frac{\partial h}{\partial \tau}\ \delta \frac{\partial h}{\partial \tau} \right\}$$

All terms are of the form

$$\frac{\partial F}{\partial \tau}\ \delta \frac{\partial F}{\partial \tau}$$

Since

$$\delta \frac{\partial F}{\partial \tau} = -\frac{\partial F}{\partial a}\frac{\partial}{\partial \tau}\delta a - \frac{\partial F}{\partial b}\frac{\partial}{\partial \tau}\delta b$$

A term of the general form

$$\int d\tau \iint da\, db \;\; \frac{\partial F}{\partial \tau}\, \delta \frac{\partial F}{\partial \tau}$$

$$= \int d\tau \iint da\, db \;\; \frac{\partial F}{\partial \tau}\left(-\frac{\partial F}{\partial a}\frac{\partial}{\partial \tau}\delta a - \frac{\partial F}{\partial b}\frac{\partial}{\partial \tau}\delta b \right)$$

$$= \int d\tau \iint da\, db \;\; \frac{\partial}{\partial \tau}\left(\frac{\partial F}{\partial \tau}\frac{\partial F}{\partial a} \right)\delta a + \frac{\partial}{\partial \tau}\left(\frac{\partial F}{\partial \tau}\frac{\partial F}{\partial b} \right)\delta b$$

$$= \int d\tau \iint da\, db \;\; -\frac{\partial}{\partial \tau}\left(\frac{\partial F}{\partial \tau}\frac{\partial F}{\partial a} \right)\frac{\partial \psi}{\partial b} + \frac{\partial}{\partial \tau}\left(\frac{\partial F}{\partial \tau}\frac{\partial F}{\partial b} \right)\frac{\partial \psi}{\partial a}$$

$$= \int d\tau \iint da\, db \;\; \frac{\partial}{\partial \tau}\frac{\partial\left(F, \dot{F}\right)}{\partial(a,b)} \;\; \psi$$

Applying this result, we obtain

$$\int d\tau \iint da\, db \left\{ \frac{\partial x}{\partial \tau}\; \delta \frac{\partial x}{\partial \tau} + \frac{\partial y}{\partial \tau}\; \delta \frac{\partial y}{\partial \tau} + \tfrac{1}{3}\frac{\partial h}{\partial \tau}\; \delta \frac{\partial h}{\partial \tau} \right\}$$

$$= \int d\tau \iint da\, db \;\; \frac{\partial}{\partial \tau}\left(\frac{\partial(x,\dot{x})}{\partial(a,b)} + \frac{\partial(y,\dot{y})}{\partial(a,b)} + \frac{1}{3}\frac{\partial\left(h,\dot{h}\right)}{\partial(a,b)} \right) \;\psi$$

which must vanish for arbitrary ψ.

This yields the conservation law

$$\frac{\partial Q}{\partial \tau} = 0$$

where

$$Q = \frac{\partial(x,u)}{\partial(a,b)} + \frac{\partial(y,v)}{\partial(a,b)} + \frac{1}{3}\frac{\partial(h,Dh/Dt)}{\partial(a,b)} = \frac{1}{h}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + \frac{1}{3}J(h,Dh/Dt)\right)$$

That is,

$$\frac{D}{Dt}\left(\frac{\zeta + \frac{1}{3}J(Dh/Dt,h)}{h}\right) = 0$$

where

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

is the relative vorticity. The corresponding shallow-water result is

$$\frac{D}{Dt}\left(\frac{\zeta}{h}\right) = 0$$

Does this result make sense?

The general, three-dimensional, Ertel theorem for constant-density flow is

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) Q_3 = 0$$

where

$$Q_3 = \left[\left(\nabla \times \mathbf{v}\right) \cdot \nabla\theta\right]$$

If the fluid moves in vertical columns, this implies

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) Q = 0$$

where

$$Q = \frac{1}{h}\int_0^h Q_3 \;\; dz$$

For columnar motion,

$$\nabla \times \mathbf{v} = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)$$

So if we choose

$$\theta = c = \frac{z}{h} H_0$$

We obtain

$$Q_3 = J(\theta, w) + \zeta \theta_z$$

Computing

$$Q = \frac{1}{h} \int_0^h dz \ \left(J(\theta, w) + \zeta \theta_z \right) = \frac{1}{h} \int_0^h dz \ \left(J\left(\frac{z}{h}, \frac{z}{h} \frac{Dh}{Dt} \right) + \zeta \frac{\partial}{\partial z} \left(\frac{z}{h} \right) \right)$$

gives the expected result.

(The non-Hamiltonian derivation of the Green-Naghdi equations gives no hint of a potential-vorticity law)

Nearly geostrophic flow

Lagrangian for the shallow water equations:

$$L\big[x(a,b,\tau),y(a,b,\tau)\big] = \tfrac{1}{2}\iint da\,db\left\{\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 - gh\right\}$$

$$h \equiv \frac{\partial(a,b)}{\partial(x,y)}$$

It will be convenient to use the *extended* form of Hamilton's principle.

So we do a quick review of this.

The Lagrangian given above is analogous to

$$\delta\int_{t_1}^{t_2} L(q_i,\dot{q}_i)\,dt = 0$$

which gives the *Euler-Lagrange equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}$$

These inspire us to define the generalized *momenta*

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

and Hamiltonian

$$H \equiv \sum_i p_i \dot{q}_i - L$$

from which it follows that

$$dH = \sum_i \left\{ \dot{q}_i \, dp_i + p_i \, d\dot{q}_i - \frac{\partial L}{\partial q_i} dq_i - \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i \right\} = \sum_i \left\{ \dot{q}_i \, dp_i - \frac{\partial L}{\partial q_i} dq_i \right\}$$

which in turn implies

$$\frac{\partial H}{\partial p_i} = \dot{q}_i, \qquad \frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}$$

Using the Euler-Lagrange equation, these are equivalent to

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = +\frac{\partial H}{\partial p_i} \qquad \text{(the \textit{canonical equations})}$$

and the variational principle

$$\delta \int_{t_1}^{t_2} dt \left\{ p_i \frac{dq_i}{dt} - H(p, q, t) \right\} = 0$$

in which

$$\delta q_i \quad \text{and} \quad \delta p_i$$

are taken independently.

For the shallow-water Lagrangian

$$L = \tfrac{1}{2}\iint da\,db\left\{\left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 - gh\right\}$$

we have:

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \quad \leftrightarrow \quad \mathbf{p}(\mathbf{a}) = \frac{\delta L}{\delta \mathbf{x}} = \frac{\partial \mathbf{x}}{\partial \tau} = \mathbf{u}$$

$$H \equiv \sum_i p_i \dot{q}_i - L \quad \leftrightarrow \quad H = \iint d\mathbf{a}\ \ \mathbf{u}\cdot\frac{\partial \mathbf{x}}{\partial \tau} - L \ = \ \tfrac{1}{2}\iint da\,db\left\{u^2 + v^2 + gh\right\}$$

$$\delta\int_{t_1}^{t_2} dt\left\{p_i\frac{dq_i}{dt} - H(p,q,t)\right\} = 0 \quad \leftrightarrow$$

$$\delta\int d\tau\left\{\iint d\mathbf{a}\ \ \mathbf{u}(\mathbf{a},\tau)\cdot\frac{\partial \mathbf{x}(\mathbf{a},\tau)}{\partial \tau} - H[\mathbf{x},\mathbf{u}]\right\} = 0$$

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = +\frac{\partial H}{\partial p_i} \quad \leftrightarrow$$

$$\frac{\partial \mathbf{u}}{\partial \tau} = -\frac{\delta H}{\delta \mathbf{x}}, \qquad \frac{\partial \mathbf{x}}{\partial \tau} = \frac{\delta H}{\delta \mathbf{u}}$$

The shallow-water Lagrangian in *rotating coordinates*

To add Coriolis force with a completely arbitrary Coriolis parameter

$$f(x,y)$$

add "potentials" $R(x,y)$ and $P(x,y)$ to the Lagrangian

$$L[\mathbf{u}(\mathbf{a},\tau),\mathbf{x}(\mathbf{a},\tau)] = \iint d\mathbf{a}\left\{(u-R)\frac{\partial x}{\partial \tau} + (v+P)\frac{\partial y}{\partial \tau}\right\} - H$$

such that

$$\frac{\partial R}{\partial y} + \frac{\partial P}{\partial x} = f(x,y)$$

The Hamiltonian

$$H = \tfrac{1}{2}\iint d\mathbf{a}\left\{u^2 + v^2 + g\frac{\partial(a,b)}{\partial(x,y)}\right\}$$

is the same as in the nonrotating case. As always

$$h = \frac{\partial(a,b)}{\partial(x,y)}$$

$$L\left[\mathbf{u}(\mathbf{a},\tau),\mathbf{x}(\mathbf{a},\tau)\right] = \iint d\mathbf{a}\left\{(u - R)\frac{\partial x}{\partial \tau} + (v + P)\frac{\partial y}{\partial \tau}\right\} - H$$

The variations yield the *rotating* shallow-water equations

$$\delta u: \quad u = \frac{\partial x}{\partial \tau}, \qquad \delta v: \quad v = \frac{\partial y}{\partial \tau}$$

$$\delta x: \quad \frac{\partial u}{\partial \tau} - f\frac{\partial y}{\partial \tau} = -g\frac{\partial h}{\partial x}, \qquad \delta y: \quad \frac{\partial v}{\partial \tau} + f\frac{\partial x}{\partial \tau} = -g\frac{\partial h}{\partial y}$$

These equations conserve the energy

$$H = \tfrac{1}{2}\iint d\mathbf{a}\left\{u^2 + v^2 + gh\right\}$$

and the potential vorticity

$$\frac{\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} + f}{h}$$

We are interested in *nearly geostrophic* flow, i.e. flow in which ε is small:

$$L[\mathbf{u},\mathbf{x}] = \iint d\mathbf{a}\left\{(\varepsilon u - R)\frac{\partial x}{\partial \tau} + (\varepsilon v + P)\frac{\partial y}{\partial \tau}\right\} - H$$

The most drastic approximation sets ε=0, corresponding to the <u>constraint</u>:

$$\mathbf{u}(\mathbf{a},\tau) = 0$$

The resulting Lagrangian

$$L_0[\mathbf{x}(\mathbf{a},\tau)] = \iint d\mathbf{a}\left\{-R(x,y)\frac{\partial x}{\partial \tau} + P(x,y)\frac{\partial y}{\partial \tau} - \tfrac{1}{2}g\frac{\partial(a,b)}{\partial(x,y)}\right\}$$

depends only on **x**, not on **u**.   The variations yield

$$\delta x: \qquad -f\frac{\partial y}{\partial \tau} = -g\frac{\partial h}{\partial x}, \qquad\qquad \delta y: \qquad +f\frac{\partial x}{\partial \tau} = -g\frac{\partial h}{\partial y}$$

Since the continuity equation is implicit, the complete dynamics are

$$-fv = -g\frac{\partial h}{\partial x}$$

$$fu = -g\frac{\partial h}{\partial y}$$

$$\frac{Dh}{Dt} + h\nabla\cdot\mathbf{u} = 0$$

called **planetary geostrophic dynamics**.

A **less drastic** approximation replaces **u** by the geostrophic velocity

$$u = u_G[\mathbf{x}(\mathbf{a},\tau)] \equiv -\frac{g}{f}\frac{\partial h}{\partial y}, \qquad v = v_G[\mathbf{x}(\mathbf{a},\tau)] \equiv \frac{g}{f}\frac{\partial h}{\partial x}$$

This corresponds to a projection in phase space
onto a manifold with half the dimensions of the full phase space.

The Lagrangian becomes

$$L_1[\mathbf{x}(\mathbf{a},\tau)] = \iint d\mathbf{a}\left\{(u_G - R)\frac{\partial x}{\partial \tau} + (v_G + P)\frac{\partial y}{\partial \tau}\right\} - H$$

with

$$H_1[\mathbf{x}(\mathbf{a},\tau)] = \tfrac{1}{2}\iint d\mathbf{a}\left\{u_G{}^2 + v_G{}^2 + g\frac{\partial(a,b)}{\partial(x,y)}\right\}$$

The dynamics

$$\delta \int d\tau \;\; L_1[\mathbf{x}(\mathbf{a},\tau)] = 0$$

yields equations with the same accuracy as the quasigeostrophic equations but without the requirement that the fluid depth be nearly uniform. The L_1 −dynamics conserves the energy H_1 and the potential vorticity

$$\frac{\dfrac{\partial v_G}{\partial x} - \dfrac{\partial u_G}{\partial y} + f}{h}$$