Eulerian Forms of Hamilton's Principle

The fluid motion is a time-dependent map

$$\mathbf{x} = \mathbf{x}(\mathbf{a},\tau)$$

from **x**-space to **a**-space.

Hamilton's principle

$$\delta \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) \right\} = 0$$

requires that the action be stationary with respect to $\delta\mathbf{x}(\mathbf{a},\tau)$. But each forward map corresponds to an inverse map

$$\mathbf{a} = \mathbf{a}(\mathbf{x},t)$$

Thus Hamilton's principle is equivalent to

$$\delta \int dt \iiint d\mathbf{x} \frac{\partial(\mathbf{a})}{\partial(\mathbf{x})} \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) \right\} = 0$$

for arbitrary $\delta\mathbf{a}(\mathbf{x},t)$.

To express

$$\mathbf{v} \equiv \frac{\partial \mathbf{x}}{\partial \tau}$$

as an **a**-derivative, we use

$$\left(\frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) a_i = 0, \qquad i = 1, 2, 3$$

Alternatively, we may treat the preceding equations as constraints. Then Hamilton's principle becomes

$$\delta \int dt \iiint d\mathbf{x} \frac{\partial(\mathbf{a})}{\partial(\mathbf{x})} \left\{ \tfrac{1}{2} \mathbf{v} \cdot \mathbf{v} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) - \mathbf{A} \cdot \frac{D\mathbf{a}}{Dt} \right\} = 0$$

for arbitrary $\delta\mathbf{a}(\mathbf{x},t)$, $\delta\mathbf{v}(\mathbf{x},t)$ and $\delta\mathbf{A}(\mathbf{x},t)$.
A is the Lagrange multiplier corresponding to the constraint that defines **v**. We choose $c=S$ for simplicity.

In the same way, we may eliminate

$$\rho = \frac{\partial(\mathbf{a})}{\partial(\mathbf{x})}$$

by attaching its time-derivative

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

as another constraint. We obtain…

$$\delta \int dt \iiint d\mathbf{x} \left\{ \tfrac{1}{2} \rho \, \mathbf{v} \cdot \mathbf{v} - \rho \, E\left(\frac{1}{\rho}, S \right) - \rho \Phi(\mathbf{x}) - \rho \mathbf{A} \cdot \frac{D\mathbf{a}}{Dt} + \phi \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) \right\} = 0$$

for arbitrary variations in

$\rho(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$, $S(\mathbf{x},t)$, $\mathbf{A}(\mathbf{x},t)$, $\mathbf{a}(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$ (with $c=S$).

However, only 3 of our 4 constraints are independent.
Therefore, one constraint may be dropped.
If we drop the B-constraint, we have

$$\delta \int dt \iiint d\mathbf{x} \left\{ \tfrac{1}{2} \rho \, \mathbf{v} \cdot \mathbf{v} - \rho \, E\left(\frac{1}{\rho}, S \right) - \rho \Phi(\mathbf{x}) - \rho A \frac{Da}{Dt} - \rho C \frac{DS}{Dt} - \rho \frac{D\phi}{Dt} \right\} = 0$$

for arbitrary variations in

$\rho(\mathbf{x},t)$, $\mathbf{v}(\mathbf{x},t)$, $S(\mathbf{x},t)$, $A(\mathbf{x},t)$, $a(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$.

There is one more thing we can do to simplify the variational principle. We use

$$\delta \mathbf{v} : \qquad \mathbf{v} = A \nabla a + C \nabla S + \nabla \phi$$

to eliminate $\mathbf{v}$ itself. After a little work, we obtain…

$$\delta \int dt \left[\iiint d\mathbf{x} \left(\rho A \frac{\partial a}{\partial t} + \rho C \frac{\partial S}{\partial t} + \rho \frac{\partial \phi}{\partial t} \right) + H \right] = 0$$

where

$$H[\rho, A, a, C, S, \phi] = \iiint d\mathbf{x} \left\{ \tfrac{1}{2} \rho \mathbf{v} \cdot \mathbf{v} + \rho E\left(\frac{1}{\rho}, S \right) + \rho \Phi(\mathbf{x}) \right\}$$

and

$$\mathbf{v} \equiv A \nabla a + C \nabla S + \nabla \phi$$

This variational principle *bears no resemblance* to what we started with! But if we have not made a mistake, it must give us the perfect-fluid equations.

To test it, we compute the variations

$$\delta A: \quad \frac{Da}{Dt} = 0, \qquad \delta a: \quad \frac{DA}{Dt} = 0$$

$$\delta C: \quad \frac{DS}{Dt} = 0, \qquad \delta \eta: \quad \frac{DC}{Dt} = \frac{\partial}{\partial S} E\left(\frac{1}{\rho}, S \right) \equiv T$$

$$\delta \phi: \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\delta \rho: \quad A \frac{\partial a}{\partial t} + C \frac{\partial S}{\partial t} + \frac{\partial \phi}{\partial t} + \tfrac{1}{2} \mathbf{v} \cdot \mathbf{v} + \Phi + E + \frac{p}{\rho} = 0$$

$$p \equiv -\frac{\partial}{\partial \alpha} E(\alpha, S)$$

Are these equations equivalent to the perfect fluid equations?

First note

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} = \nabla A \times \nabla a + \nabla C \times \nabla S$$

so

$$\boldsymbol{\omega} \times \mathbf{v} = (\mathbf{v} \cdot \nabla A)\nabla a - (\mathbf{v} \cdot \nabla a)\nabla A + (\mathbf{v} \cdot \nabla C)\nabla S - (\mathbf{v} \cdot \nabla S)\nabla C$$

Then

$$\begin{aligned}
\frac{\partial \mathbf{v}}{\partial t} &= \frac{\partial A}{\partial t}\nabla a + \frac{\partial C}{\partial t}\nabla S + A\nabla\frac{\partial a}{\partial t} + C\nabla\frac{\partial S}{\partial t} + \nabla\frac{\partial \phi}{\partial t} \\
&= \frac{\partial A}{\partial t}\nabla a + \frac{\partial C}{\partial t}\nabla S - \frac{\partial a}{\partial t}\nabla A - \frac{\partial S}{\partial t}\nabla C + \nabla\left(A\frac{\partial a}{\partial t} + C\frac{\partial S}{\partial t} + \frac{\partial \phi}{\partial t}\right) \\
&= -(\mathbf{v} \cdot \nabla A)\nabla a - (\mathbf{v} \cdot \nabla C - T)\nabla S + (\mathbf{v} \cdot \nabla a)\nabla A + (\mathbf{v} \cdot \nabla S)\nabla C \\
&\qquad - \nabla\left(\tfrac{1}{2}\mathbf{v} \cdot \mathbf{v} + \Phi + E + \frac{p}{\rho}\right)
\end{aligned}$$

which is equivalent to

$$\frac{\partial \mathbf{v}}{\partial t} = -(\boldsymbol{\omega} \times \mathbf{v}) - \frac{1}{\rho}\nabla p - \nabla\Phi - \nabla\left(\tfrac{1}{2}\mathbf{v} \cdot \mathbf{v}\right).$$

QED

What has happened to the particle-relabeling symmetry?

It is present as a *gauge symmetry*.

In the Hamiltonian

$$H[\rho, A, a, C, S, \phi] = \iiint d\mathbf{x} \left\{ \tfrac{1}{2} \rho \, \mathbf{v} \cdot \mathbf{v} + \rho \, E\left(\frac{1}{\rho}, S \right) + \rho \Phi(\mathbf{x}) \right\}$$

with $\quad \mathbf{v} \equiv A \nabla a + C \nabla S + \nabla \phi$

The *four* potentials

$$A, \; a, \; C, \; \phi$$

appear only in the *three* components

$$u, \; v, \; w$$

of **v**. Therefore, it is possible to vary the four potentials in a way that is not detected by the Hamiltonian. This leads to Ertel's theorem.

Flows with special symmetry

Setting a=A=0 in the general form of Hamilton's principle reduces it to:

$$\delta\int dt\left[\iiint d\mathbf{x}\left(\rho C\frac{\partial S}{\partial t}+\rho\frac{\partial\phi}{\partial t}\right)+H\right]=0$$

where

$$H[\rho,C,S,\phi]=\iiint d\mathbf{x}\left\{\tfrac{1}{2}\rho\,\mathbf{v}\cdot\mathbf{v}+\rho\,E\left(\frac{1}{\rho},S\right)+\rho\Phi(\mathbf{x})\right\}$$

and

$$\mathbf{v}\equiv C\nabla S+\nabla\phi$$

Solutions of the results equations are a *subset* of the set of general solutions to the perfect fluid equations; they have vanishing circulation

$$\oint\mathbf{v}\cdot d\mathbf{x}=0$$

on isentropic surfaces. If the flow is homentropic we may also set S=C=0. Then the whole dynamics reduces to the variational principle

$$\delta\int dt\iiint d\mathbf{x}\;\;\rho\left\{\frac{\partial\phi}{\partial t}+\tfrac{1}{2}\nabla\phi\cdot\nabla\phi+E\left(\frac{1}{\rho}\right)+\Phi(\mathbf{x})\right\}=0$$

for irrotational flow

$$\mathbf{v}=\nabla\phi$$

Poisson bracket formulation

Return temporarily to the case of discrete variables. The canonical equations are:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dq_i}{dt} = +\frac{\partial H}{\partial p_i}, \qquad i = 1,2,\ldots,N$$

Define the Poisson bracket:

$$\{A,B\} \equiv \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i} \right)$$

Then the canonical equations take the form:

$$\frac{dp_i}{dt} = \{p_i, H\}, \qquad \frac{dq_i}{dt} = \{q_i, H\}$$

More generally,

$$\frac{dF}{dt} = \{F,H\}$$

for any F.

Thus the whole dynamics has just two ingredients:

1. The Hamiltonian H, a scalar function.

2. The Poisson bracket, a bilinear operator.

These two objects are called *geometrical objects* because they have important properties that survive transformation to new variables.

Coordinate transformations

If $$z \equiv \left(z^1, z^2, \ldots, z^{2N}\right) \equiv \left(q_1, q_2, \ldots, q_N, p_1, p_2, \ldots, p_N\right)$$

The canonical equations take the form

$$\frac{dz^i}{dt} = J^{ij}\frac{\partial H}{\partial z^j}$$

where

$$J = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix}$$

The Poisson bracket takes the form

$$\{A, B\} \equiv \frac{\partial A}{\partial z^i} J^{ij} \frac{\partial B}{\partial z^j}$$

The equations * are covariant with respect to coordinate transformations

$$\bar{z}^i = \bar{z}^i(z)$$

That is
$$\{A, B\} = \frac{\partial A}{\partial \bar{z}^m} \bar{J}^{mn} \frac{\partial B}{\partial \bar{z}^n}$$

if J obeys the transformation rule for a contravariant tensor:

$$\bar{J}^{mn} = \frac{\partial \bar{z}^m}{\partial z^i} J^{ij} \frac{\partial \bar{z}^n}{\partial z^j}$$

Geometrical properties

The symplectic tensor J has the following properties

1. nonsingularity: $\det\left(J^{ij}\right) \neq 0$

2. antisymmetry: $J^{ij} = -J^{ji}$

3. Jacobi property: $J^{im}\dfrac{\partial J^{jk}}{\partial z^m} + J^{jm}\dfrac{\partial J^{ki}}{\partial z^m} + J^{km}\dfrac{\partial J^{ij}}{\partial z^m} = 0$

These properties are called *geometric properties*, because they hold in any system of coordinates.

To see this, realize that these 3 properties are trivially satisfied in canonical coordinates, and that the properties themselves are covariant. The first property holds in the new coordinates only if the coordinate transformation is itself nonsingular:

$$\det\left(\frac{\partial \bar{z}^i}{\partial z^j}\right) \neq 0$$

In coordinate-free notation these same 3 properties may be written:

1. nonsingularity: $\{A, B\} \neq 0$

2. antisymmetry: $\{A,B\} = -\{B, A\}$

3. Jacobi property $\{A,\{B,C\}\} + \{B,\{C, A\}\} + \{C,\{A,B\}\} = 0$

General definition of a Hamiltonian system

A **Hamiltonian system** consists of a scalar function *H* and a Poisson bracket obeying the 3 properties above.

(The nonsingularity property is sometimes omitted with interesting consequences.)

Example: Poisson bracket for irrotational flow

Recall:

$$\delta \int dt \left\{ \iiint d\mathbf{x} \;\; \phi \frac{\partial \rho}{\partial t} - H \right\} = 0$$

where

$$H = \iiint d\mathbf{x} \;\; \rho \left\{ \tfrac{1}{2} \nabla\phi \cdot \nabla\phi + E\left(\frac{1}{\rho}\right) + \Phi(\mathbf{x}) \right\}$$

This is in canonical form (in Eulerian variables).

Therefore

$$\{A,B\} = \iiint d\mathbf{x} \left(\frac{\delta A}{\delta \rho} \frac{\delta B}{\delta \phi} - \frac{\delta A}{\delta \phi} \frac{\delta B}{\delta \rho} \right)$$

Check:

$$\begin{aligned} \frac{\partial}{\partial t} \rho(\mathbf{x}_0) &= \{\rho(\mathbf{x}_0), H\} \\ &= \iiint d\mathbf{x} \left(\frac{\delta \rho(\mathbf{x}_0)}{\delta \rho(\mathbf{x})} \frac{\delta H}{\delta \phi(\mathbf{x})} \right) \\ &= \iiint d\mathbf{x} \;\; \delta(\mathbf{x} - \mathbf{x}_0) \; [-\nabla \cdot (\rho \nabla \phi)] \\ &= [-\nabla \cdot (\rho \mathbf{v})] \Big|_{\mathbf{x} = \mathbf{x}_0} \end{aligned}$$

QED

Example: Perfect fluid in one dimension.

This too takes the canonical form, but in Lagrangian coordinates.

Recall:

$$\delta \int d\tau \left\{ \int da \;\; u(a,\tau) \frac{\partial x(a,\tau)}{\partial \tau} \;\; - H \right\} = 0$$

where

$$H\big[u(a), x(a)\big] = \int da \left\{ \tfrac{1}{2} u(a)^2 + E\left(\frac{dx}{da}\right) + \Phi\big(x(a)\big) \right\}$$

Thus

$$\{A, B\} = \int da \left(\frac{\delta A}{\delta x(a)} \frac{\delta B}{\delta u(a)} - \frac{\delta A}{\delta u(a)} \frac{\delta B}{\delta x(a)} \right)$$

The dynamics is

$$\frac{dF}{dt} = \{F, H\}$$

Check:

$$\frac{\partial x(a,\tau)}{\partial \tau} = \{x(a),H\} = \int da' \left(\frac{\delta x(a)}{\delta x(a')} \frac{\delta H}{\delta u(a')} - \frac{\delta x(a)}{\delta u(a')} \frac{\delta H}{\delta x(a')} \right)$$

$$= \int da' \left(\frac{\delta x(a)}{\delta x(a')} \frac{\delta H}{\delta u(a')} \right)$$

Using

$$x(a) = \int da' \; x(a')\delta(a-a') \qquad \Rightarrow \qquad \frac{\delta x(a)}{\delta x(a')} = \delta(a-a')$$

and

$$\frac{\delta H}{\delta u(a')} = u(a')$$

we obtain

$$\frac{\partial x(a,\tau)}{\partial \tau} = \int da' \; \delta(a-a')u(a') = u(a,\tau)$$

Similarly

$$\frac{\partial u(a,\tau)}{\partial \tau} = \{u(a),H\} = \int da' \left(-\frac{\delta u(a)}{\delta u(a')} \frac{\delta H}{\delta x(a')} \right) = -\frac{dx}{da}\frac{\partial p}{\partial x} - \frac{\partial \Phi}{\partial x}$$

Example: one dimensional homentropic fluid in Eulerian variables

The method will be to transform the bracket

$$\{A,B\} = \int da \left(\frac{\delta A}{\delta x(a)} \frac{\delta B}{\delta u(a)} - \frac{\delta A}{\delta u(a)} \frac{\delta B}{\delta x(a)} \right)$$

from Lagrangian coordinates

$$x(a,\tau),\ \ u(a,\tau)$$

to Eulerian coordinates

$$u(x,t),\ \ \rho(x,t)$$

Motivation: The Hamiltonian takes the simplest form in Eulerian variables.

We use the chain rule for functional derivatives:

$$\frac{\delta A}{\delta x(a)} = \int dx' \left\{ \frac{\delta A}{\delta u(x')} \frac{\delta u(x')}{\delta x(a)} + \frac{\delta A}{\delta \rho(x')} \frac{\delta \rho(x')}{\delta x(a)} \right\}$$

$$\frac{\delta A}{\delta u(a)} = \int dx' \left\{ \frac{\delta A}{\delta u(x')} \frac{\delta u(x')}{\delta u(a)} + \frac{\delta A}{\delta \rho(x')} \frac{\delta \rho(x')}{\delta u(a)} \right\}$$

To calculate the needed derivatives, write:

$$u(x') = \int da\ u(a)\delta(a - a')$$

Thus

$$\frac{\delta u(x')}{\delta x(a)} = 0 \quad \text{and} \quad \frac{\delta u(x')}{\delta u(a)} = \delta(a - a')$$

Similarly

$$\rho(x') = \int dx\ \rho(x)\delta(x - x') = \int da\ \ \delta(x(a) - x(a'))$$

implies

$$\frac{\delta\rho(x')}{\delta x(a)} = \delta'(x - x') \quad \text{and} \quad \frac{\delta\rho(x')}{\delta u(a)} = 0$$

Collecting results

$$\frac{\delta A}{\delta x(a)} = \int dx' \frac{\delta A}{\delta\rho(x')}\delta'(x - x') = \frac{\partial}{\partial x}\left(\frac{\delta A}{\delta\rho(x)}\right)$$

and

$$\frac{\delta A}{\delta u(a)} = \int dx' \frac{\delta A}{\delta u(x')}\delta(a - a') = \int da' \frac{1}{\rho'}\frac{\delta A}{\delta u(x')}\delta(a - a') = \frac{1}{\rho}\frac{\delta A}{\delta u(x)}$$

Therefore, finally,

$$\{A, B\} = \int dx \left(\frac{\partial}{\partial x}\left(\frac{\delta A}{\delta\rho}\right)\frac{\delta B}{\delta u} - \frac{\partial}{\partial x}\left(\frac{\delta B}{\delta\rho}\right)\frac{\delta A}{\delta u}\right)$$

Final result for 1d homentropic fluid

The dynamics is

$$\frac{dF}{dt} = \{F, H\}$$

where

$$\{A, B\} = \int dx \left(\frac{\partial}{\partial x}\left(\frac{\delta A}{\delta \rho}\right)\frac{\delta B}{\delta u} - \frac{\partial}{\partial x}\left(\frac{\delta B}{\delta \rho}\right)\frac{\delta A}{\delta u}\right)$$

and

$$H = \int dx\ \rho(x)\left(\tfrac{1}{2}u(x)^2 + E\left(\frac{1}{\rho(x)}\right) + \Phi(x)\right)$$

Check:

$$\frac{\delta H}{\delta \rho(x)} = \tfrac{1}{2}u^2 + E + \Phi(x) - \frac{1}{\rho}E\,'$$

$$\frac{\delta H}{\delta u(x)} = \rho u$$

$$\frac{\delta \rho(x)}{\delta \rho(x')} = \delta(x - x') \qquad \text{and} \qquad \frac{\delta \rho(x)}{\delta u(x')} = 0$$

$$\frac{\partial}{\partial t}\rho(x) = \{\rho, H\} = \int dx' \left(\frac{\partial}{\partial x'}\big(\delta(x - x')\big)\,\rho u(x')\right) = -\frac{\partial}{\partial x}(\rho u) \qquad \text{OK}$$

General 3d perfect fluid

$$\{A,B\} = \iiint d\mathbf{x}\ [\ \nabla\left(\frac{\delta A}{\delta \rho}\right)\cdot\frac{\delta B}{\delta \mathbf{v}} - \nabla\left(\frac{\delta B}{\delta \rho}\right)\cdot\frac{\delta A}{\delta \mathbf{v}} + \frac{\nabla\times\mathbf{v}}{\rho}\cdot\frac{\delta A}{\delta \mathbf{v}}\times\frac{\delta B}{\delta \mathbf{v}}$$
$$+\frac{\nabla S}{\rho}\cdot\left(\frac{\delta A}{\delta \mathbf{v}}\frac{\delta B}{\delta S} - \frac{\delta B}{\delta \mathbf{v}}\frac{\delta A}{\delta S}\right)\ ]$$

(Morrison and Greene)

This result is somewhat tedious to work out by transforming from the canonical form (as we just did for the 1d case).

In fact, sometimes it is better to *guess* the Poisson bracket, and then verify your guess a posteriori.

Example of guessing:

Poisson bracket for the quasigeostrophic equation

$$\frac{\partial}{\partial t}\nabla^2\psi + J\left(\psi, \nabla^2\psi + h(x,y)\right) = 0$$

states that the potential vorticity

$$q = \zeta + h = \nabla^2\psi + h$$

Let

$$A[q]$$

be any functional of q. Then

$$\frac{dA}{dt} = \iint dx\,dy\,\frac{\delta A}{\delta\zeta}\frac{\partial\zeta}{\partial t} = -\iint dx\,dy\,\frac{\delta A}{\delta\zeta}J(\psi, \zeta + h)$$
$$= \iint dx\,dy\;q\,J\left(\psi, \frac{\delta A}{\delta\zeta}\right)$$

On the other hand

$$H = \iint dx\,dy\;\tfrac{1}{2}\nabla\psi\cdot\nabla\psi$$

implies that

$$\delta H = \iint dx\,dy\;\nabla\psi\cdot\nabla\delta\psi = -\iint dx\,dy\;\psi\,\delta\zeta$$

Hence

$$\frac{\delta H}{\delta \zeta} = -\psi$$

and our evolution equation takes the form

$$\frac{dA}{dt} = \iint dx\,dy \;\; q \; J\left(\frac{\delta A}{\delta \zeta}, \frac{\delta H}{\delta \zeta}\right)$$

This fits the Hamiltonian form

$$\frac{dA}{dt} = \{A, H\}$$

if

$$\{A, B\} = \iint dx\,dy \;\; q \; J\left(\frac{\delta A}{\delta \zeta}, \frac{\delta B}{\delta \zeta}\right)$$

This bracket is obviously symmetric. With some work, it can be shown to obey the Jacobi identity. However, like almost all Eulerian brackets it is singular. In fact,

$$\{A, C\} = 0$$

for all C of the form

$$C = \iint dx\,dy \;\; F(q)$$

Much more about singular Poisson brackets!

Interesting fact

The quasigeostrophic bracket is not unique.

We may use

$$\frac{dA}{dt} = \{A, H\}$$

with

$$H = \iint dx\, dy\ \tfrac{1}{2} \nabla\psi \cdot \nabla\psi$$

and

$$\{A, B\} = \iint dx\, dy\ \ q\ J\left(\frac{\delta A}{\delta\zeta}, \frac{\delta B}{\delta\zeta}\right)$$

OR

$$\frac{dA}{dt} = \{A, Z\}$$

with

$$Z = \iint dx\, dy\ \ \tfrac{1}{2} q^2$$

and

$$\{A, B\} = \iint dx\, dy\ \ \psi\ J\left(\frac{\delta A}{\delta\zeta}, \frac{\delta B}{\delta\zeta}\right)$$