Planetary interiors:
Magnetic fields, Convection and Dynamo Theory
3. How planetary magnetic fields are generated

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Section 3.

3. How planetary magnetic fields are generated

3.1 Fundamentals of MHD and Maxwell's equations

Pre-Maxwell equations 1.

Maxwell's equations are the basis of electromagnetic theory. Foundation of dynamo theory. They are (3.1.1)-(3.1.4).

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \mu \mathbf{j} + \left[\frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} \right] \qquad (3.1.1, 3.1.2)$$

(3.1.1) is the differential form of Faraday's law of induction. If the magnetic field varies with time an electric field is produced. In an electrically conducting body, electric field drives a current. Basis of dynamo action.

(3.1.2) is Ampère's law, which relates the electric current to the magnetic field it produces.

Maxwell added the term in red, but it is not needed in MHD, because fluid flows are much slower than the speed of light.

E electric field, **B** the magnetic field, **j** is the current density, μ is the permeability.

Pre-Maxwell equations 2.

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \cdot \mathbf{E} = \frac{\rho_c}{\epsilon} \qquad (3.1.3, 3.1.4)$$

(3.1.3) expresses that there are no magnetic monopoles.

(3.1.4) says that electric field is generated by charges. However, we eliminate **E** in MHD.

E electric field, **B** the magnetic field,

ρ_c is the charge density, ϵ is the permittivity.

These equations are valid in a frame at rest.

Ohm's law

In a moving frame $\mathbf{E}$ must be replaced by $\mathbf{E} + \mathbf{u} \times \mathbf{B}$ while $\mathbf{j}$ stays the same. So in MHD Ohm's law is

$$\mathbf{j} = \sigma(\mathbf{E} + \mathbf{u} \times \mathbf{B}). \tag{3.1.5}$$

The SI unit of electrical conductivity is Siemens/metre.
The term $\mathbf{u} \times \mathbf{B}$ is critical. Flow interacting with magnetic field generates current, which in turn gives field. With the right configuration, flow can sustain field.
Define magnetic diffusivity

$$\eta = \frac{1}{\mu\sigma}, \tag{3.1.6}$$

dimensions metre^2/second.

Induction equation

Dividing Ohm's law (3.1.5) by σ and taking the curl

$$\nabla \times (\frac{\mathbf{j}}{\sigma}) = \nabla \times \mathbf{E} + \nabla \times (\mathbf{u} \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}), \quad (3.1.7)$$

and using (3.1.2) to eliminate **j**,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) - \nabla \times \eta(\nabla \times \mathbf{B}), \qquad (3.1.8)$$

remembering (3.1.6). (3.1.8) is the induction equation.

Alternative forms of the induction equation

If the conductivity is constant we can use the vector identity
curl curl = grad div -del^2
and (3.1.8) to write the constant conductivity induction equation as

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}. \tag{3.1.9}$$

An alternative form of the constant diffusivity induction equation for incompressible flow, $\nabla \cdot \mathbf{u} = 0$ is

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}. \tag{3.1.10}$$

Magnetic diffusion

If fluid velocity is zero

$$\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}. \tag{3.1.11}$$

This is diffusion equation, so if field is zero at infinity, it just diffuses away. Otherwise it comes to a uniform value.
Timescale: $\mathbf{B} = (B_0 \sin kx + B_1, 0, 0)$ in Cartesian coordinates at $t = 0$, field evolves as

$$\mathbf{B} = (B_0 \sin kx \exp(-\eta k^2 t) + B_1, 0, 0) \tag{3.1.12}$$

so e-folding time is $1/k^2\eta$. If $k = \pi/L$, e-folding time is $L^2/\pi^2\eta$.
$\eta \approx 1\,\mathrm{m}^2/\mathrm{s}$ in the Earth's core. If L is ten metres, field decays by a factor e in about 10 seconds. $L = 3.5 \times 10^6$ metres, time is 20,000 years.

Perfect Conductor

The induction equation is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}), \tag{3.1.13}$$

in the zero diffusion limit. Frozen flux limit. The flux through a closed curve C in the fluid, enclosing area A, is

$$\int_A \mathbf{B} \cdot d\mathbf{s}.$$

Alfvén's theorem says flux through any closed loop C is constant as the loop moves with the fluid.
Note that the size of the loop varies in time. If it shrinks, this means the field strength must be increasing, since magnetic flux = area times field strength.
Stretching therefore increases field intensity. Stretching is key to dynamo action.

Magnetic Reynolds number

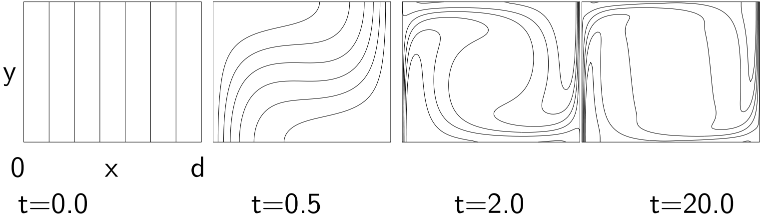
Non-dimensionalise induction equation. We choose typical length scale L_* and a typical fluid velocity U_*. Introduce scaled ˜ variables

$$t = \frac{L_*}{U_*}\tilde{t}, \qquad \mathbf{x} = L_*\tilde{\mathbf{x}}, \qquad \mathbf{u} = U_*\tilde{\mathbf{u}} \tag{3.1.14}$$

so that $\nabla = \tilde{\nabla}/L_*$, and (3.1.9) becomes

$$\frac{\partial \mathbf{B}}{\partial \tilde{t}} = \tilde{\nabla}\times(\tilde{\mathbf{u}}\times\mathbf{B}) + Rm^{-1}\tilde{\nabla}^2\mathbf{B}, \qquad Rm = \frac{U_* L_*}{\eta}, \quad (3.1.15, 3.1.16)$$

Rm being the dimensionless magnetic Reynolds number. Large Rm means induction dominates over diffusion. In astrophysics and geophysics Rm is almost always large, but in laboratory experiments it is usually small, though values up to ~ 50 can be reached in large liquid sodium facilities.



An imposed flow that resembles convection rolls

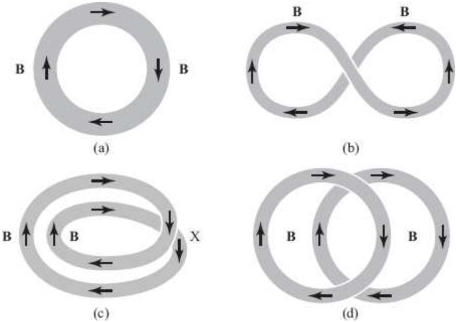
$$\mathbf{u} = (-U \sin\frac{\pi x}{d} \cos\frac{\pi y}{d}, U \cos\frac{\pi x}{d} \sin\frac{\pi y}{d}, 0) \qquad (3.1.17)$$

is imposed. The induction equation

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + Rm^{-1}\nabla^2 \mathbf{B}, \qquad (3.1.18)$$

with $Rm = 1000$ is solved numerically.
The field is initially vertical, and the field at multiples of the turn-over time d/U is shown. An initially uniform field stirred by convection quickly expels the magnetic field out of the roll.

Stretch Twist Fold dynamo



A loop of flux is first stretched to twice its length, reducing cross-section by half. By Alfvéns theorem, the field strength must double. Now twist the loop to get to (b), and then fold to get to (c). Apply small diffusion at X to reconnect. We have doubled the total flux.

3.2 Kinematic dynamo problem

Kinematic Dynamo problem

Velocity **u** is a given function of space and possibly time. Dynamic, or self-consistent, dynamo problem is when **u** is solved using the momentum equation. In kinematic dynamos, only the induction equation (3.1.9) is solved.

Kinematic dynamo problem linear in **B**. If **u** independent of time,

$$\mathbf{B} = \mathbf{B}_0(x, y, z)e^{pt}, \qquad \mathbf{B}_0 \to 0 \text{ as } \mathbf{x} \to \infty. \tag{3.2.1}$$

Usually infinite set of eigenmodes $\mathbf{B}_0$ each with a complex eigenvalue

$$p = \sigma + i\omega. \tag{3.2.2}$$

σ is the growth rate, and ω the frequency. If one or more modes have σ positive, we have a dynamo. If $\omega = 0$, steady dynamo. If $\omega \neq 0$ growing dynamo waves.

Anti-Dynamo theorem 1.

In Cartesian coordinates (x, y, z) no field independent of z which vanishes at infinity can be maintained by dynamo action. So its impossible to generate a 2D dynamo field.
Because $\nabla \cdot \mathbf{B} = 0$, any 2D field can be written

$$\mathbf{B} = B\hat{z} + \nabla \times A\hat{z}. \tag{3.2.3}$$

Insert this into the induction equation (3.1.10) and we get two equations

$$\frac{\partial A}{\partial t} + (\mathbf{u} \cdot \nabla)A = \eta \nabla^2 A, \tag{3.2.4}$$

$$\frac{\partial B}{\partial t} + (\mathbf{u} \cdot \nabla)B = \eta \nabla^2 B + \mathbf{B}_H \cdot \nabla u_z. \tag{3.2.5}$$

Anti-Dynamo theorem 1 continued

Now multiply (3.2.4) by A and integrate over the whole volume,

$$\frac{\partial}{\partial t}\int \frac{1}{2}A^2\,dv + \int \nabla\cdot\frac{1}{2}\mathbf{u}A^2\,dv = -\eta\int(\nabla A)^2\,dv. \qquad (3.2.6)$$

Divergence term vanishes, because the fields is small at large distance.

The term on the right is negative definite, so the integral of A^2 continually decays.

It will only stop decaying if A is constant, in which case there is no field.

Decay of B

Once A has decayed to zero, $\mathbf{B}_H$ is zero, so there is no source term in (3.2.5). We can then apply the same argument to show B decays to zero. This shows that no nontrivial field 2D can be maintained as a steady (or oscillatory) dynamo.

Note that if A has very long wavelength components, it may take a very long time for A to decay to zero, and in that time B might grow quite large as a result of the driving by the last term in (3.2.5). But ultimately it must decay.

Anti-Dynamo theorem 2

No dynamo can be maintained by a planar flow

$$\mathbf{u} = (u_x(x, y, z, t), u_y(x, y, z, t), 0)$$

No restriction is placed on whether the field is 2D or not in this theorem.

The z-component of (3.1.10) is

$$\frac{\partial B_z}{\partial t} + \mathbf{u} \cdot \nabla B_z = \eta \nabla^2 B_z, \tag{3.2.7}$$

because the $\mathbf{B} \cdot \nabla u_z$ is zero because u_z is zero. Multiplying (3.2.7) by B_z and integrating, again the advection term gives a surface integral vanishing at infinity, and so B_z decays.
If $B_z = 0$, then

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0. \tag{3.2.8}$$

which means

$$B_x = \frac{\partial A}{\partial y}, \qquad B_y = -\frac{\partial A}{\partial x} \tag{3.2.9}$$

for some A, and then the z-component of the curl of the induction equation gives

$$\frac{\partial \nabla_H^2 A}{\partial t} + \nabla_H^2(\mathbf{u} \cdot \nabla A) = \eta \nabla_H^2 \nabla^2 A, \quad \nabla_H^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{3.2.10}$$

Fourier transform in x and y, the $\nabla_H^2 = k_x^2 + k_y^2$ which then can be cancelled out, so (3.2.10) is just (3.2.4) again, which on multiplying through by A leads to the decay of A again. So the whole field decays if the flow is planar.

Axisymmetric field decomposition

An axisymmetric field and flow can be decomposed into

$$\mathbf{B} = B\hat{\phi} + \mathbf{B}_P = B\hat{\phi} + \nabla \times A\hat{\phi}, \quad \mathbf{u} = s\Omega\hat{\phi} + \mathbf{u}_P = s\Omega\hat{\phi} + \nabla \times \frac{\psi}{s}\hat{\phi},$$

$$s = r\sin\theta. \tag{3.2.11}$$

The induction equation now becomes quite simple,

$$\frac{\partial A}{\partial t} + \frac{1}{s}(\mathbf{u}_P \cdot \nabla)(sA) = \eta(\nabla^2 - \frac{1}{s^2})A, \tag{3.2.12}$$

$$\frac{\partial B}{\partial t} + s(\mathbf{u}_P \cdot \nabla)(\frac{B}{s}) = \eta(\nabla^2 - \frac{1}{s^2})B + s\mathbf{B}_P \cdot \nabla\Omega. \tag{3.2.13}$$

Gives important insight into the dynamo process.

Anti-Dynamo theorem 3, Cowling's theorem

An axisymmetric magnetic field vanishing at infinity cannot be maintained by dynamo action.
Polar coordinate version of theorem 1. Multiplying (3.2.12) by $s^2 A$ and integrating, and eliminating the divergence terms by converting them to surface integrals which vanish at infinity, we get

$$\frac{\partial}{\partial t} \int \frac{1}{2} s^2 A^2 \, dv = -\eta \int |\nabla (sA)|^2 \, dv \tag{3.2.14}$$

which shows that sA decays.

Note that $sA = \textit{constant}$ would give diverging A at $s = 0$ which is not allowed.

Cowling's theorem continued

Once A has decayed, $\mathbf{B}_P = 0$ in (3.2.14), and now multiplying (3.2.14) by B/s^2 gives

$$\frac{\partial}{\partial t} \int \frac{1}{2} s^{-2} B^2 \, dv = -\eta \int |\nabla(\frac{B}{s})|^2 \, dv \qquad (3.2.15)$$

and since we don't allow B proportional to s, which doesn't vanish at infinity, this shows that B must decay also. So there can be no steady axisymmetric dynamo.

Note this theorem disallows axisymmetric $\mathbf{B}$, not axisymmetric $\mathbf{u}$. The Ponomarenko dynamo and the Dudley and James dynamos (see section 2 below) are working dynamos with axisymmetric $\mathbf{u}$ but with nonaxisymmetric $\mathbf{B}$.

Anti-Dynamo theorem 4, Toroidal flow theorem

A purely toroidal flow, that is one with $\mathbf{u} = \nabla \times T\mathbf{r}$ for some scalar function T, cannot maintain a dynamo. Note that this means that there is no radial motion, $u_r = 0$.
This is the polar coordinate version of theorem 2. First we show the radial component of field decays, because

$$\frac{\partial}{\partial t}(\mathbf{r} \cdot \mathbf{B}) + \mathbf{u} \cdot \nabla(\mathbf{r} \cdot \mathbf{B}) = \eta \nabla^2 (\mathbf{r} \cdot \mathbf{B}), \tag{3.2.16}$$

so multiplying through by $(\mathbf{r} \cdot \mathbf{B})$ and integrating does the job. Then a similar argument to that used to prove theorem 2 shows that the toroidal field has no source term and so decays. For details see Gilbert (2003) p 380. It is not necessary to assume either flow or field is axisymmetric for this theorem.

3.3 Working kinematic dynamos

Kinematic Dynamos

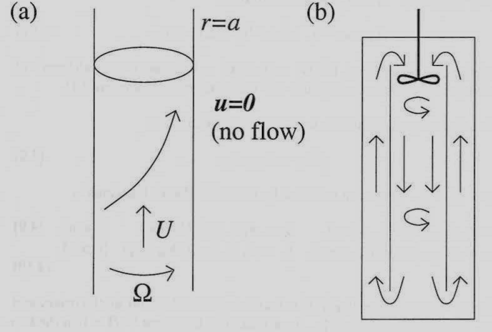
We look at dynamos with a prescribed velocity field, mostly a time-independent velocity

Ponomarenko dynamo: simplest known kinematic dynamo, model for Riga dynamo experiment

G.O.Roberts dynamo: model for the Karlsruhe experiment. Also has roll geometry, and a field on scale larger than roll size

Spherical dynamos: the Dudley-James dynamo.

Ponomarenko Dynamo



(a) Ponomarenko flow,

$$\mathbf{u} = s\Omega\hat{\phi} + U\hat{\mathbf{z}},\ \ s < a;\ \ \mathbf{u} = 0,\ \ s > a \qquad (3.3.1)$$

Solid body screw motion inside cylinder $s = a$.

(b) Riga dynamo experiment configuration

Flow has helicity,

$$H = \mathbf{u} \cdot \nabla \times \mathbf{u} = \mathbf{u} \cdot \zeta = U \frac{1}{s} \frac{\partial}{\partial s} s^2 \Omega = 2U\Omega. \qquad (3.3.2)$$

Discontinuity of $\mathbf{u}$ at $s = a$ provides strong shearing.
Evades planar motion anti-dynamo theorem through U.

$$Rm = \frac{a\sqrt{U^2 + a^2\Omega^2}}{\eta}$$

based on maximum velocity.
Seek nonaxisymmetric field of form

$$\mathbf{B} = \mathbf{b}(s) \exp[(s + i\omega)t + im\phi + ikz] \qquad (3.3.3)$$

thus evading Cowling's theorem. Induction equation, (3.1.10),

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \eta \nabla^2 \mathbf{B}.$$

Ponomarenko Dynamo 3. Equations

$$(\mathbf{u}\cdot\nabla)\mathbf{B} = (ikU + im\Omega)\mathbf{B} - \Omega B_\phi \hat{\mathbf{s}} + \Omega B_s \hat{\phi}, \qquad (3.3.4)$$

$$(\mathbf{B}\cdot\nabla)\mathbf{u} = -\Omega B_\phi \hat{\mathbf{s}} + \Omega B_s \hat{\phi}, \qquad (3.3.5)$$

so

$$p^2 b_s = \Delta_m b_s - \frac{2im}{s^2} b_\phi, \quad p^2 b_\phi = \Delta_m b_\phi + \frac{2im}{s^2} b_s, \quad (3.3.6, 3.3.7)$$

where

$$\Delta_m = \frac{1}{s}\frac{\partial}{\partial s} s \frac{\partial}{\partial s} - \frac{1}{s^2} - \frac{m^2}{s^2}$$

Inside, $s < a$, $p = p_i$, $\eta p_i^2 = \sigma + i\omega + im\Omega + ikU + \eta k^2$,
Outside, $s > a$, $p = p_e$, $\eta p_e^2 = \sigma + i\omega + \eta k^2$.
Defining $b_\pm = b_s \pm i b_\phi$,

$$p^2 b_\pm = \Delta_{m\pm 1} b_\pm. \qquad (3.3.8)$$

Solutions are

$$b_\pm = A_\pm \frac{I_{m\pm1}(p_i s)}{I_{m\pm1}(p_i a)},\ \ s < a, \qquad A_\pm \frac{K_{m\pm1}(p_i s)}{K_{m\pm1}(p_i a)}, s > a. \quad (3.3.9)$$

I_m and K_m are the modified Bessel functions (like sinh and cosh) that are zero at $s = 0$ and zero as $s \to \infty$ respectively.
With this choice, the fields are continuous at $s = a$. Also need E_z continuous (1.4.8d), so $\eta(\nabla \times \mathbf{B})_z - u_\phi B_s$ has to be continuous, using (3.1.5), giving

$$\eta \left(\frac{\partial b_\phi}{\partial s}|_{s\to a+} - \frac{\partial b_\phi}{\partial s}|_{s\to a-} \right) = a\Omega b_s(a) \qquad (3.3.10)$$

writing the jump as [.],

$$2\eta \left[\frac{\partial b_\pm}{\partial s} \right] = \pm i a\Omega (b_+(a) + b_-(a)). \qquad (3.3.11)$$

Defining

$$S_{\pm} = \frac{p_i I'_{m\pm1}(p_i a)}{I_{m\pm1}(p_i a)} - \frac{p_e K'_{m\pm1}(p_i a)}{K_{m\pm1}(p_e a)} \qquad (3.3.12)$$

the dispersion relation is

$$2\eta S_+ S_- = ia\Omega(S_+ - S_-). \qquad (3.3.13)$$

Needs a simple MATLAB code to sort it out.
Non-dimensionalise on length scale a and timescale a^2/η and dimensionless parameters are growth-rate s/Ω, frequency ω/Ω, pitch of spiral $\chi = U/a\Omega$, ka and m. The diffusion coefficient $\eta/a^2\Omega = (1+\chi^2)^{1/2} Rm^{-1}$.
For marginal stability set $s = 0$. For given χ, ka and m adjust Rm and ω until real and imaginary parts of $2\eta S_+ S_- - ia\Omega(S_+ - S_-) = 0$. Minimise Rm over m and ka to get the critical mode, and over χ to get the optimum pitch angle.

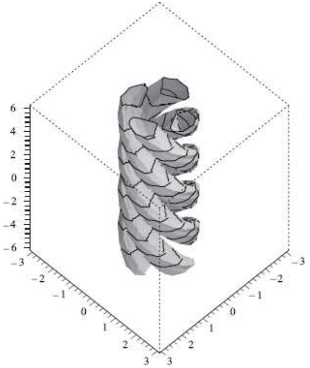
When all this is done, we find $Rm_{crit} = 17.7221$, $ka = -0.3875$, $m = 1$, $a^2\omega/\eta = -0.4103$ and $\chi = 1.3141$. Poloidal and toroidal flow similar.

This is a low value of Rm, which motivated the Riga dynamo experiment. Magnetic field is strongest near $s = a$ where it is generated by shear.

At large Rm, there is a significant simplification, because then $m\Omega + kU$ is small, so $p_e = p_i$ and the ηk^2 terms are small. Bessel functions have asymptotic simplifications at large argument.

Fastest growing modes given by

$$|m| = (6(1+\chi^{-2}))^{-3/4}\left(\frac{a^2\Omega}{2\eta}\right)^{1/2}, \quad s = 6^{-3/2}\Omega(1+\chi^{-2})^{-1/2}. \tag{3.3.14}$$



Magnetic field for the Ponomarenko dynamo at large *Rm*. Surface of constant **B** shows spiralling field following flow spiral, and located near the discontinuity.

The Ponomarenko dynamo has a single roll, and the field at low *Rm* is on scale of the roll, smaller at high *Rm*.

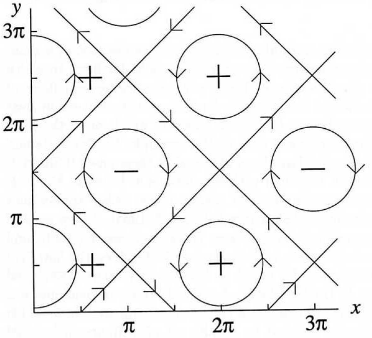
G.O Roberts dynamo has a collection of rolls and field can be coherent across many rolls.

Special case of ABC flows (Arno'ld, Beltrami, Childress)

$$\mathbf{u} = (C\sin z + B\cos y, A\sin x + C\cos z, B\sin y + A\cos x) \quad (3.3.15)$$

with $A = B = 1,\ C = 0$. These flows have $\nabla \times \mathbf{u} = \mathbf{u}$, so vorticity = velocity.

Clearly ABC flows have helicity.



Flow is two dimensional, independent of z, but has a z-component.

$$\mathbf{u} = (\cos y, \sin x, \sin y + \cos x). \qquad (3.3.16)$$

Avoids the planar antidynamo theorem.

G.O.Roberts dynamo 3.

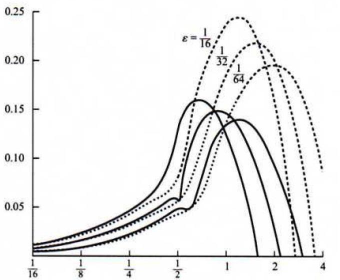
$$\mathbf{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, \psi \right), \qquad \psi = \sin y + \cos x. \tag{3.3.17}$$

Magnetic field has to be z-dependent (Anti-dynamo theorem 1)

$$\mathbf{B} = \mathbf{b}(x, y) \exp(pt + ikz). \tag{3.3.18}$$

$\mathbf{b}(x, y)$ is periodic in x and y, but it has a mean part independent of x and y which spirals in the z-direction.
To solve the problem, Roberts inserted the form of $\mathbf{B}$ into the induction equation, using a double Fourier series expansion of $\mathbf{b}(x, y)$.
The coefficients then form a linear matrix eigenvalue problem for p.

p



k

Growth rate p as a function of z wavenumber, k for various $\epsilon = Rm^{-1}$.

Solid lines: G.O. Roberts numerical results. Dashed lines, A.M. Soward's asymptotic large Rm theory.

Large Rm G.O.Roberts dynamo

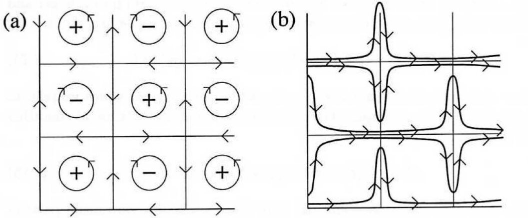


Figure rotated through 45°. At large Rm, generated field is expelled into boundary layers.
This gives enhanced diffusion, leading to lower growth rates and ulitimately to decay.
$p \rightarrow 0$ as $Rm \rightarrow \infty$ means dynamo is slow.

Following Bullard and Gellman, 1954, velocity for kinematic spherical dynamos written

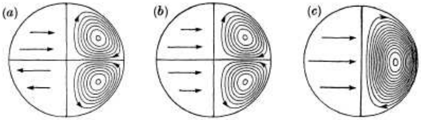
$$\mathbf{u} = \sum_i \mathbf{t}_l^m + \mathbf{s}_l^m \tag{3.3.19}$$

where $\mathbf{t}_l^m$ and $\mathbf{s}_l^m$ are the toroidal and poloidal components

$$\mathbf{t}_l^m = \nabla \times \hat{\mathbf{r}} t_l^m(r,t) Y_l^m(\theta,\phi), \qquad \mathbf{s}_l^m = \nabla \times \nabla \times \hat{\mathbf{r}} s_l^m(r,t) Y_l^m(\theta,\phi) \tag{3.3.20}$$

where $-l \leq m \leq l$.
Bullard and Gellman used $\mathbf{u} = \epsilon \mathbf{t}_1^0 + \mathbf{s}_2^2$ with $t_1^0(r) = r^2(1-r)$, $s_2^2(r) = r^3(1-r)^2$. In their original calculations, they found dynamo action, but subsequent high resolution computations showed they were not dynamos. Warning: inadequate resolution can lead to bogus dynamos!

Dudley and James dynamos



Dudley and James looked at 3 models, using the notation of (3.3.20),

$$\mathbf{u} = \mathbf{t}_2^0 + \epsilon \mathbf{s}_2^0,\ (a); \quad \mathbf{u} = \mathbf{t}_1^0 + \epsilon \mathbf{s}_2^0,\ (b); \quad \mathbf{u} = \mathbf{t}_1^0 + \epsilon \mathbf{s}_1^0,\ (c) \quad (3.3.21)$$

with

$$t_1^0 = s_1^0 = r \sin \pi r, \qquad t_2^0 = s_2^0 = r^2 \sin \pi r.$$

All steady axisymmetric flows.

Dudley and James dynamos 2.

t components give azimuthal flow only, s components give meridional flow.

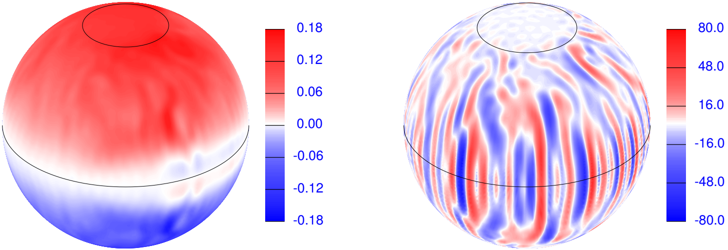
All three models give dynamo action. Since the flow is axisymmetric, the field has $\exp im\phi$ dependence, and $m = 1$ is preferred.

(a) with $\epsilon = 0.14$ has $Rm_{crit} \approx 54$ (steady). (b) with $\epsilon = 0.13$ has $Rm_{crit} \approx 95$ (oscillatory). (c) with $\epsilon = 0.17$ has $Rm_{crit} \approx 155$ (oscillatory).

In all cases, the toroidal and meridional flows are of similar magnitude. Field is basically an equatorial dipole, which in oscillatory cases rotates in time.

3.4 Field generation in geodynamo models

Field generation in numerical dynamos

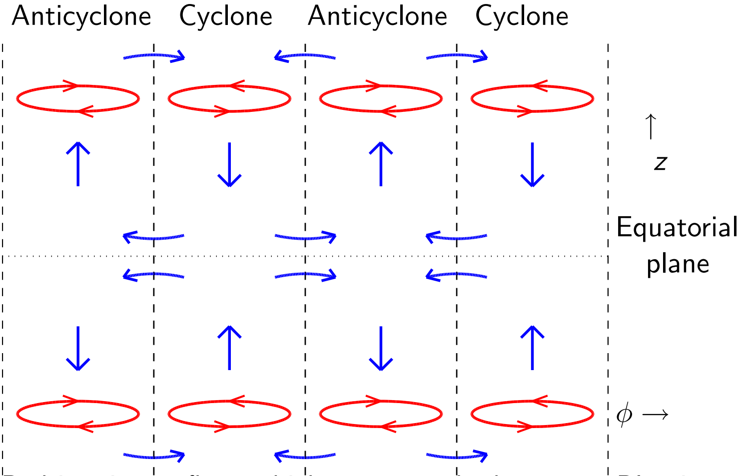


$E = 5 \times 10^{-5}$, $Pr = Pm = 1$, $Ra = 400$ with stress-free boundaries.

Left: radial magnetic field at the CMB. Right: radial velocity at $r = r_i + 0.8$.

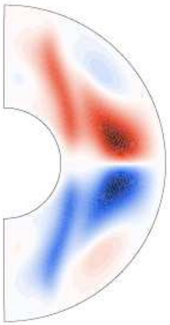
This is a standard dipolar solution which persists for all time. Flow has the columns predicted by linear theory, though they are now time-dependent.

Flow in the convection columns



Red is primary flow, which convects the heat out. Blue is secondary flow which provides helicity $\mathbf{u} \cdot \zeta$, where ζ is the vorticity. Helicity is important for **dynamo action**.

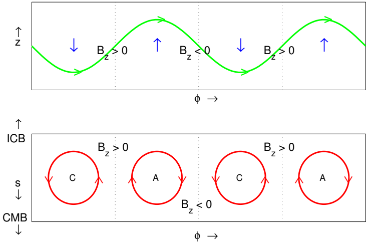
B_ϕ from dynamo model



$< B_\phi >$

Azimuthal average of B_ϕ for a moderately supercritical dynamo Antisymmetric about the equator, so $B_\phi = 0$ on the equator. Note $B_z > 0$ near tangent cylinder, usually gives $B_\phi > 0$ in N. hemisphere. Field generally increases with s near ICB.

Start with $B_\phi > 0$ in northern hemisphere where $u_z > 0$ in anticyclones, A, $u_z < 0$ in cyclones, C.

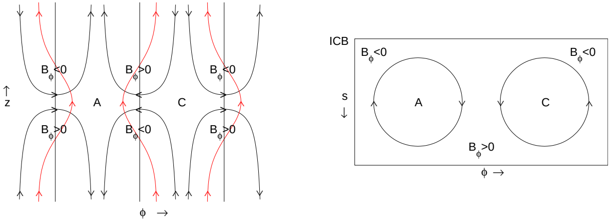


Green magnetic field line initially in ϕ-direction is displaced parallel to rotation axis z by secondary flow.

Primary flow sweeps positive B_z towards the ICB, negative B_z towards the CMB.

Net effect is to create positive B_z inside, negative B_z outside, just as in a dipolar field. In S. hemisphere B_ϕ and u_z are reversed, so effect is same.

Mechanism for generating B_ϕ from B_z



Northern Hemisphere. $u_z > 0$ in anticyclones, A, $u_z < 0$ in cyclones, C.
Left: u_ϕ flow resulting from u_z profile stretches out red positive B_z to give B_ϕ.
Right: constant z section in Northern Hemisphere viewed from above.
Positive B_ϕ is moved outward in radius, negative B_ϕ moved inward in radius by the vortex circulations. This reinforces the original B_ϕ configuration and allows the magnetic field to grow.

3.5 Fast and slow dynamos

If magnetic diffusion time $\gg$ turn-over time, (3.1.10) becomes

$$\frac{\partial \mathbf{B}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{B} = \mathbf{B} \cdot \nabla \mathbf{u} + \epsilon \nabla^2 \mathbf{B}. \tag{3.5.1}$$

where $\epsilon = Rm^{-1}$ is small. Time scaled on turnover time L/U.
For steady flow $\mathbf{B} \sim e^{\sigma t}$, and if $\gamma = Re(\sigma)$ flow is a fast dynamo if

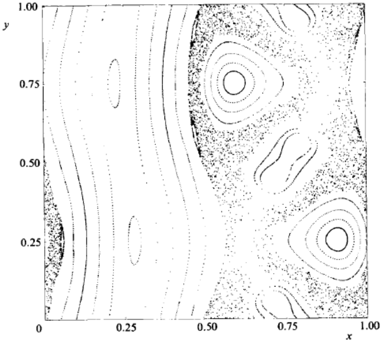
$$\gamma_0 = \lim_{\epsilon \to 0} \gamma(\epsilon) > 0. \tag{3.5.2}$$

Flow is a slow dynamo if

$$\gamma_0 = \lim_{\epsilon \to 0} \gamma(\epsilon) \leq 0. \tag{3.5.3}$$

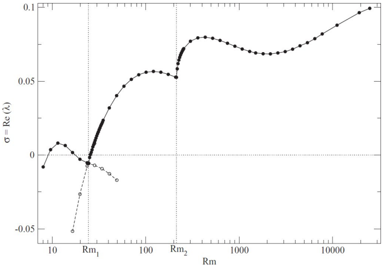
Fast dynamos grow on the turnover time (months in the Sun) not the magnetic diffusion time (millions of years in the Sun).

ABC flow



Typical Poincaré section shows chaotic regions and ordered regions. Ordered regions called KAM regions, or KAM tori. 'Normal' in chaotic ODEs. The $ABC = 1$ flow is unusual in having rather large KAM regions and small chaotic regions.

ABC dynamo results



Dynamo growth rate γ against $\log_{10} Rm$.
Because the chaotic regions which give stretching are small, quite difficult to show its a fast dynamo numerically.
However, γ seems to be increasing gradually rather than decreasing at large Rm.

Stretching properties

Take a point **a** and a small vector **v** origin at **a**. Now integrate the particle path ODE's with initial conditions $\mathbf{x_1} = \mathbf{a}$ and $\mathbf{x_2} = \mathbf{a} + \mathbf{v}$, and monitor $d = |\mathbf{x_1} - \mathbf{x_2}|$.
If stretching is occuring, d will grow exponentially.
Liapunov exponent is

$$\Lambda(\mathbf{a}) = \max_{\mathbf{v}} \lim_{t\to\infty} \sup \frac{\ln d}{t} \tag{3.5.4}$$

The maximum Liapunov exponent is found by taking the supremum over all **a**.
Can be computed, but expensive! A practical definition of chaos is that the Liapunov exponent is positive. In a given chaotic region, Λ is usually the same, but it is zero in KAM regions.

Rather than have a fully 3D steady flow field, like $A = B = C = 1$, we can choose a 2D flow but make it a time-dependent flow. Field still has $\exp ikz$ dependence, which makes numerics a lot easier.

Galloway-Proctor CP flow:

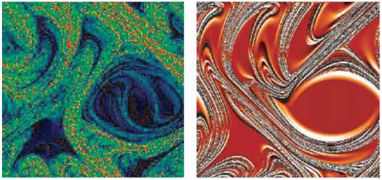
$$\mathbf{u} = \nabla\times(\psi(x, y, t)\hat{\mathbf{z}})+\gamma\psi(x, y, t)\hat{\mathbf{z}}, \quad \psi = \sin(y+\sin t)+\cos(x+\cos t) \tag{3.5.5}$$

Very like the G.O. Roberts flow, except the stagnation point pattern rotates round in a circle.

Also an LP flow, with $\psi = \sin(y + \cos t) + \cos(x + \cos t)$.

Flow is now non-integrable, and has positive Liapunov exponents.

Galloway-Proctor dynamo

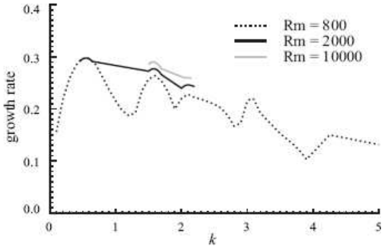


CP flow

Left: Liapunov exponents: blue regions have little or no stretching, green/red has order one stretching

Right: snapshot of the normal field B_z. Note the good correlation between strong field and strong stretching.

Galloway-Proctor dynamo results



CP flow.
Growth rate against wavenumber k, $\exp ikz$, for $Rm = 800$, $Rm = 2,000$ and $Rm = 10,000$.

Note that the growth rate continues to increase as Rm is increased, unlike the G.O. Roberts flow results.

3.6 Mean field dynamo theory

About Mean Field Dynamo Theory

Subject divides into two areas

(i) Underlying theory of MFDT, conditions for its validity, its relationship to turbulence theory and its extension to include nonlinear effects

(ii) The solutions of MFDT equations and the new types of dynamos they create: dynamo waves, $\alpha\omega$ dynamos and α^2 dynamos.

There is surprisingly little interaction between these two activities. Vastly more papers have been written on (ii), almost all accepting the MFDT equations as a useful model.

Mean and Fluctuating parts. 1.

Basic idea is to split the magnetic field and the flow into mean and fluctuating parts,

$$\mathbf{B} = \overline{\mathbf{B}} + \mathbf{B}', \qquad \mathbf{u} = \overline{\mathbf{u}} + \mathbf{u}' \tag{3.6.1}$$

Reynolds averaging rules: assume a linear averaging process

$$\overline{\mathbf{B_1} + \mathbf{B_2}} = \overline{\mathbf{B}}_1 + \overline{\mathbf{B}}_2, \qquad \overline{\mathbf{u_1} + \mathbf{u_2}} = \overline{\mathbf{u}}_1 + \overline{\mathbf{u}}_2 \tag{3.6.2}$$

and once its averaged it stays averaged, so

$$\overline{\overline{\mathbf{B}}} = \overline{\mathbf{B}}, \qquad \overline{\overline{\mathbf{u}}} = \overline{\mathbf{u}}, \tag{3.6.3}$$

So averaging

$$\overline{\mathbf{B}'} = \overline{\mathbf{u}'} = 0. \tag{3.6.4}$$

Also, assume averaging commutes with differentiating, so

$$\overline{\frac{\partial \mathbf{B}}{\partial t}} = \frac{\partial}{\partial t}\overline{\mathbf{B}}, \qquad \overline{\nabla \mathbf{B}} = \nabla \overline{\mathbf{B}}. \tag{3.6.5}$$

Now we average the induction equation

$$\overline{\frac{\partial \mathbf{B}}{\partial t}} = \overline{\nabla \times (\mathbf{u} \times \mathbf{B})} + \eta \overline{\nabla^2 \mathbf{B}},$$

Using the Reynolds averaging rules

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times \overline{(\mathbf{u} \times \mathbf{B})} + \eta \nabla^2 \overline{\mathbf{B}}. \tag{3.6.6}$$

The interesting term is $\overline{(\mathbf{u} \times \mathbf{B})}$.

$$\overline{\mathbf{u} \times \mathbf{B}} = \overline{(\overline{\mathbf{u}} + \mathbf{u}') \times (\overline{\mathbf{B}} + \mathbf{B}')} = \overline{\overline{\mathbf{u}} \times \overline{\mathbf{B}} + \overline{\mathbf{u}} \times \mathbf{B}' + \mathbf{u}' \times \overline{\mathbf{B}} + \mathbf{u}' \times \mathbf{B}'}$$

Mean Field Induction equation

So we have

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \nabla \times \mathcal{E} + \eta \nabla^2 \overline{\mathbf{B}}, \qquad \mathcal{E} = \overline{\mathbf{u}' \times \mathbf{B}'}. \quad (3.6.7)$$

$\mathcal{E}$ is called the mean e.m.f. and it is a new term in the induction equation. We usually think of the primed quantities as being small scale turbulent fluctuations, and this new term comes about because the average mean e.m.f. can be nonzero if the turbulence has suitable averaged properties.

No longer does Cowling's theorem apply! With this new term, we can have simple axisymmetric dynamos. Not surprisingly, most authors have included this term in their dynamo work, though actually it can be hard to justify the new term in astrophysical applications.

If we subtract the mean field equation from the full equation,

$$\frac{\partial \mathbf{B}'}{\partial t} = \nabla\times(\overline{\mathbf{u}}\times\mathbf{B}')+\nabla\times(\mathbf{u}'\times\overline{\mathbf{B}})+\nabla\times\mathcal{G}+\eta\nabla^2\mathbf{B}',\ \mathcal{G} = \mathbf{u}'\times\mathbf{B}'-\overline{\mathbf{u}'\times\mathbf{B}'}. \tag{3.6.8}$$

This is a linear equation for $\mathbf{B}'$, with a forcing term $\nabla \times (\mathbf{u}' \times \overline{\mathbf{B}})$. $\mathbf{B}'$ can therefore be thought of as the turbulent field generated by the turbulent $\mathbf{u}'$ acting on the mean $\overline{\mathbf{B}}$. We can therefore plausibly write

$$\mathcal{E}_i = a_{ij}\overline{\mathbf{B}}_j + b_{ijk}\frac{\partial \overline{\mathbf{B}}_j}{\partial x_k} + \cdots . \tag{3.6.9}$$

where the tensors a_{ij} and b_{ijk} depend on $\mathbf{u}'$ and $\overline{\mathbf{u}}$.

Mean Field Dynamo equations

We don't know $\mathbf{u}'$ and its unobservable, so we assume a_{ij} and b_{ijk} are simple isotropic tensors

$$a_{ij} = \alpha(\mathbf{x})\delta_{ij}, \qquad b_{ijk} = -\beta(\mathbf{x})\epsilon_{ijk}. \tag{3.6.10}$$

We now have the mean field dynamo theory (MFDT) equations in usual form,

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\bar{\mathbf{u}} \times \overline{\mathbf{B}}) + \nabla \times \alpha \overline{\mathbf{B}} - \nabla \times (\beta \nabla \times \overline{\mathbf{B}}) + \eta \nabla^2 \overline{\mathbf{B}}. \tag{3.6.11}$$

If β is constant, $\nabla \times (\beta \nabla \times \overline{\mathbf{B}}) = -\beta \nabla^2 \overline{\mathbf{B}}$ so the β term acts like an enhanced diffusivity. Even if it isn't constant, we recall from (3.1.8) that the term has the same form as the molecular diffusion term.

We can now justify taking a large diffusion, choosing it to give agreement with observation.

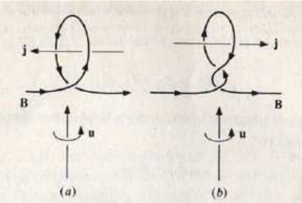
Parker loop mechanism

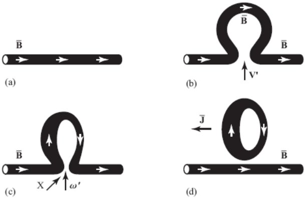
Mean field theory predicts an e.m.f. parallel to the mean magnetic field.

$$\frac{\partial \overline{\mathbf{B}}}{\partial t} = \nabla \times (\overline{\mathbf{u}} \times \overline{\mathbf{B}}) + \nabla \times \alpha \overline{\mathbf{B}} + \eta_T \nabla^2 \overline{\mathbf{B}}. \qquad (3.6.12).$$

Contrast with $\mathbf{u} \times \mathbf{B}$ perpendicular to the e.m.f. With constant α, the α-effect predicts growth of field parallel to current $\mu \nabla \times \mathbf{B}$.

Recalling that the α-effect depends on helicity, we can picture this process.





A rising twisting element of fluid brings up magnetic field. A loop of flux is created, which then twists due to helicity. The loop current is parallel to the original mean field. Poloidal field has been created out of azimuthal field.

Note that if too much twist, current in opposite direction. First order smoothing assumes small twist.

Joy's law

A sunspot pair is created when an azimuthal loop rises through solar photosphere. The vertical field impedes convection producing the spot.

Joy's law says that sunspot pairs are systematically tilted, with the leading spot being nearer the equator. Assuming flux was created as azimuthal flux deep down, suggests that loop has indeed twisted through a few degrees as it rose.

Provides some evidence of the α-effect at work. Basis of many solar dynamo models.

3.7 Mean field α-effect dynamos

Mean field spherical dynamo equations with isotropic α are

$$\frac{\partial A}{\partial t} + \frac{1}{s}(\mathbf{u}_P \cdot \nabla)(sA) = \alpha B + \eta(\nabla^2 - \frac{1}{s^2})A, \qquad (3.7.1)$$

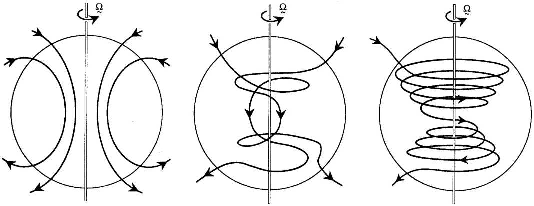
$$\frac{\partial B}{\partial t} + s(\mathbf{u}_P \cdot \nabla)(\frac{B}{s}) = \nabla \times \alpha \mathbf{B}_P + \eta(\nabla^2 - \frac{1}{s^2})B + s\mathbf{B}_P \cdot \nabla\Omega. \qquad (3.7.2)$$

The α-effect term is the source for generating poloidal field from azimuthal field, as envisaged by Parker, and Babcock & Leighton.

Two ways of generating azimuthal field from poloidal field: the α-effect or the ω effect. If first dominates, its an α^2-dynamo. If the second its an $\alpha\omega$ dynamo.

There are also $\alpha^2\omega$ dynamos where both mechanisms operate.

The Omega-effect



An initial loop of meridional field threads through the sphere.

Inside of sphere is rotating faster than outside: differential rotation.

The induction term $s\mathbf{B}_P \cdot \nabla\Omega$ generates azimuthal field by stretching. Opposite sign B_ϕ across equator as in Sun.

Dynamo waves 1.

Cartesian geometry, independent of y.

$$\mathbf{B} = (-\partial A/\partial z, B, \partial A/\partial x), \qquad \mathbf{u} = (-\partial\psi/\partial z, u_y, \partial\psi/\partial x), \tag{3.7.3}$$

$$\frac{\partial A}{\partial t} + \frac{\partial(\psi, A)}{\partial(x,z)} = \alpha B + \eta\nabla^2 A, \tag{3.7.4}$$

$$\frac{\partial B}{\partial t} + \frac{\partial(\psi, B)}{\partial(x,z)} = \frac{\partial(A, u_y)}{\partial(x,z)} - \nabla\cdot(\alpha\nabla A) + \eta\nabla^2 B. \tag{3.7.5}$$

Set $\psi = 0$, α constant, $u_y = U'z$, a constant shear, ignore α term in B equation ($\alpha\omega$ model) and set $A = \exp(\sigma t + i\mathbf{k}\cdot\mathbf{x})$. Dispersion relation is

$$(\sigma + \eta k^2)^2 = ik_x\alpha U'. \tag{3.7.6}$$

Dynamo waves 2.

Giving

$$\sigma = \frac{1+i}{\sqrt{2}}(\alpha U' k_x)^{1/2} - \eta k^2 \qquad (3.7.7)$$

with suitable choice of signs. This gives growing dynamo waves if $\alpha U'$ term overcomes diffusion.

If the wave is confined to a plane layer, $k_z = \pi/d$ gives the lowest critical mode, and there a critical value of k_x for dynamo action. The dimensionless combination $D = \alpha U' d^3/\eta^2$ is called the dynamo number,and in confined geometry there is a critical D for onset.

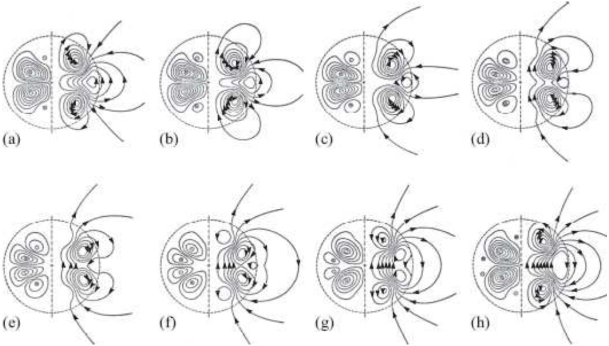
Note fastest growing waves have $k_z = 0$ so propagate perpendicular to shear direction z. If $\alpha U' > 0$, +ve k_x give growing modes with $Im(\sigma) > 0$, so propagate in -ve x-direction. -ve k_x has growing modes with $Im(\sigma) < 0$, so waves always propagate in -ve x-direction. Direction waves travel in depends on sign of $\alpha U'$.

Now set $\psi = u_y = 0$, α constant, $A = \exp(\sigma t + i\mathbf{k}\cdot\mathbf{x})$ to get dispersion relation

$$(\sigma + \eta k^2)^2 = \alpha^2 k^2$$

$$\sigma = \pm\alpha k - \eta k^2 \qquad (3.7.8)$$

which has growing modes with zero frequency. No dynamo waves, but a steady dynamo results.
In bounded geometry there is a critical α for dynamo action.



Dipolar oscillatory solution of axisymmetric $\alpha\omega$-dynamo in a sphere. (a)-(h) goes through one period. Right meridional field, left azimuthal field. $\alpha = f(r)\cos\theta$, $\omega = \omega(r)$. B antisymmetric about equator, A symmetric.