

Because the plume is accelerated by buoyancy, the momentum flux changes with z as

$$\frac{\partial M}{\partial z} = \pi b^2 g', \quad (9)$$

$$= \frac{(\pi g' b^2 w)/2 \cdot \pi b^2 w}{(\pi b^2 w^2)/2}, \quad (10)$$

$$= \frac{FQ}{M}. \quad (11)$$

We can set $F = F_0$ (constant) because the entrainment not only decrease the buoyancy g' but also increase the flow flux Q . Therefore, the equations of change of Q and M are written as

$$\frac{\partial Q}{\partial z} = 2\sqrt{2}\pi^{\frac{1}{2}}\alpha M^{\frac{1}{2}}, \quad (12)$$

$$\frac{\partial M}{\partial z} = \frac{F_0 Q}{M}. \quad (13)$$

Now, we assume Q and M as

$$Q = Q_0(z + z_V)^q, \quad (14)$$

$$M = M_0(z + z_V)^m, \quad (15)$$

where z_V is the height of “vertical origin”, and Q_0, M_0 are constants. This assumption means that the radius of the plume b increase linearly from 0 at its vertical origin $z = -z_V$. Then (12) and (13) become

$$Q_0 q (z + z_V)^{q-1} = 2\sqrt{2}\pi^{\frac{1}{2}}\alpha M_0^{\frac{1}{2}} (z + z_V)^{\frac{m}{2}}, \quad (16)$$

$$M_0 m (z + z_V)^{m-1} = \frac{F_0 Q_0}{M_0} (z + z_V)^{q-m}. \quad (17)$$

From above equations, we obtain

$$q = \frac{5}{3}, \quad (18)$$

$$m = \frac{4}{3}. \quad (19)$$

Substituting (18) and 19) into (16) and (17) with some algebra, we obtain

$$M_0 = \left(\frac{9\sqrt{2}\pi^{\frac{1}{2}}\alpha F_0}{10} \right)^{\frac{2}{3}}, \quad (20)$$

$$Q_0 = \frac{6\pi^{\frac{1}{2}}\alpha}{5} \left(\frac{18\pi^{\frac{1}{2}}\alpha F_0}{5} \right)^{\frac{1}{3}}. \quad (21)$$